

SISSA



ISAS

Area of Mathematics

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

MODULI SPACES OF FRAMED SHEAVES ON STACKY ALE SPACES, DEFORMED PARTITION FUNCTIONS AND THE AGT CONJECTURE

Supervisors

Prof. Ugo Bruzzo
Dr. Francesco Sala

Candidate

Mattia Pedrini

Submitted in partial fulfillment of the
requirements for the SISSA
degree of "Doctor Philosophiæ"

Academic Year 2012/2013

Contents

Introduction	v
Conventions and notations	xvii
Chapter 1. Projective, root and toric stacks	1
1.1. Projective stacks	1
1.2. Root stacks	8
1.3. Toric varieties	11
1.4. Picard stacks and Deligne-Mumford tori	12
1.5. Toric Deligne-Mumford stacks	15
1.6. Toric Deligne-Mumford stacks and stacky fans	19
Chapter 2. Infinite dimensional Lie algebras and representation theory	25
2.1. Heisenberg algebras	25
2.2. Affine Kac-Moody algebras of type \hat{A}_{k-1}	27
2.3. Frenkel-Kac construction	31
Chapter 3. Moduli of framed sheaves on projective stacks	33
3.1. Framed sheaves on projective stacks	33
3.2. Construction of the moduli spaces	36
3.3. $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on two-dimensional projective orbifolds	44
3.4. $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on two-dimensional projective root toric orbifolds	51
Chapter 4. ALE spaces and root stack compactification	57
4.1. Singularities of toric surfaces and their resolutions	57
4.2. Minimal resolution of $\mathbb{C}^2/\mathbb{Z}_k$	60
4.3. Stacky compactifications of X_k	64
4.4. Characterization of the stacky divisors	71
Chapter 5. Supersymmetric gauge theories on ALE spaces	79
5.1. Moduli spaces of framed sheaves on \mathcal{X}_k	79
5.2. Torus action and tangent bundle	84
5.3. $\mathcal{N} = 2$ pure gauge theories	93
5.4. $\mathcal{N} = 2$ gauge theories with one adjoint hypermultiplet	102
Chapter 6. AGT conjecture for $U(1)$ -gauge theories on \mathbb{R}^4	105
6.1. Preliminaries	105
6.2. Equivariant cohomology of $\text{Hilb}^n(\mathbb{C}^2)$	108
6.3. $\mathcal{N} = 2$ $U(1)$ -gauge theory on \mathbb{R}^4	112
Chapter 7. AGT conjecture for $U(1)$ -gauge theories on ALE spaces	119

7.1.	Setting and statement of the result	119
7.2.	The equivariant cohomology of $\text{Hilb}^n(X_k)$	122
7.3.	The basic representation of $\mathcal{A}(1, k)$	127
7.4.	$\mathcal{N} = 2$ $U(1)$ gauge theory on X_k	132
Appendix A. Serre duality for smooth projective stacks		137
Appendix B. Töen-Riemann-Roch theorem		141
Appendix C. The dimension of the moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$		143
3.1.	The inertia stack of \mathcal{X}_k	143
3.2.	Topological invariants of \mathcal{X}_k and \mathcal{D}_∞	147
3.3.	The computation of the Euler characteristic	147
3.4.	Dimension formula	153
Appendix D. The edge contribution		155
4.1.	Generalities	155
4.2.	The induction	156
4.3.	Characters of the restrictions and final results	158
Appendix E. Vanishing theorems for tautological line bundles		163
Appendix F. Identities on complex roots of unity		167
6.1.	Identities for the B contributions	167
6.2.	Identities for C and D contributions	168
Bibliography		175

Introduction

In this thesis we present a new algebro-geometric approach to the study of gauge theories on ALE spaces, which uses the theory of sheaves on *toric stacks*. This approach allows us to show the connection with gauge theories on \mathbb{R}^4 and to extend the relation with representation theory, via an AGT type relation for ALE spaces. We construct a *stacky* compactification of the minimal resolution $X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$, that is, a projective toric orbifold $\mathcal{X}_k = X_k \cup \mathcal{D}_\infty$ in which \mathcal{D}_∞ is a μ_k -gerbe. We apply here the theory of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves developed in [23], in order to obtain a moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ parameterizing $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaves on \mathcal{X}_k . We show that this moduli space actually is a smooth quasi-projective variety, and we compute its dimension. We define an analog of the deformed Nekrasov partition function ([88]) and we explicitly compute it, obtaining expressions for its *instanton part* and *deformed instanton part*, both for pure $\mathcal{N} = 2$ $U(r)$ -gauge theories and for $U(r)$ -gauge theories with one adjoint hypermultiplet. The form of the expressions give us blowup-type formulae for these partition functions, relating them with the corresponding Nekrasov partition functions on the open affine toric subsets of X_k . Finally we focus on $U(1)$ gauge theories, for which we state and prove an AGT-type relation for the pure and adjoint hypermultiplet case.

The results we present here are part of two joint work in progress, one with U. Bruzzo, F. Sala and R. Szabo [22] and one with F. Sala and R. Szabo [96].

Hystorical background. In [34] Donaldson proved that gauge-equivalence classes of framed $SU(r)$ -instantons with instanton number n on \mathbb{R}^4 are in one-to-one correspondence with isomorphism classes of locally free sheaves on \mathbb{CP}^2 of rank r and second Chern class n that are trivial along a fixed line l_∞ . The corresponding moduli space $\mathcal{M}^{reg}(r, n)$ parameterizes the isomorphism classes of the pairs (E, ϕ_E) , where E is the holomorphic bundle, and ϕ_E its trivialization on l_∞ ; the morphism ϕ_E is called a *framing at infinity* for E . More generally, one can allow E to be a torsion-free coherent sheaf on \mathbb{CP}^2 ; the corresponding moduli space $\mathcal{M}(r, n)$ is a nonsingular quasi-projective variety of dimension $2rn$, which contains $\mathcal{M}^{reg}(r, n)$ as an open dense subset. Because of their connections with moduli spaces of framed instantons, in the last ten years moduli spaces of framed sheaves on the complex projective plane have been studied quite extensively, e.g., they are the basis for the so-called *instanton counting*; let us briefly introduce this notion. In 1994 N. Seiberg formulated an ansatz for the exact prepotential of $\mathcal{N} = 2$ Yang-Mills theory in four dimensions with gauge group $SU(2)$. This solution has been extended to $SU(r)$. The prepotential \mathcal{F} can be decomposed as a sum of its *perturbative part* \mathcal{F}^{pert} and its *instanton part* \mathcal{F}^{inst} . In [88] Nekrasov conjectured an explicit way to compute \mathcal{F}^{inst} for $SU(r)$ -gauge theories on \mathbb{R}^4 by means of $SU(r)$ instantons with *instanton charge* n . The complete calculation using the localization formula and Young diagram combinatorics was done in [24].

$\mathcal{M}(r, n)$ represents the natural ambient to consider Nekrasov's conjecture, which we now explain. Let T_e be the maximal torus of $\mathrm{GL}(\mathbb{C}, r)$ consisting of diagonal matrices and let $T := \mathbb{C}^* \times \mathbb{C}^* \times T_e$. Define an action on $\mathcal{M}(r, n)$ as follows: for a framed sheaf (E, ϕ_E) one acts on E by pull-back with respect to the action of two fixed nonzero complex numbers (t_1, t_2) on \mathbb{CP}^2 and on ϕ_E by multiplication by a diagonal matrix $\mathrm{diag}(e_1, e_2, \dots, e_r)$ of order r . For $k = 1, \dots, r$, let e_k be the 1-dimensional T -module given by $(t_1, t_2, e_1, \dots, e_r) \mapsto e_k$. In the same way consider the 1-dimensional T -modules t_1, t_2 . Let $\varepsilon_1, \varepsilon_2$ and a_k be the first Chern classes of t_1, t_2 and e_k , $k = 1, 2, \dots, r$. Thus the T -equivariant cohomology of a point is $\mathbb{C}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]$. The *instanton part* of the Nekrasov partition function (in the following "instanton partition function" for brevity) for a $\mathcal{N} = 2$ pure $SU(r)$ -gauge theory on \mathbb{R}^4 is the generating function

$$\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}(\varepsilon_1, \varepsilon_2, a_1, \dots, a_r; \mathbf{q}) := \sum_{n=0}^{\infty} \mathbf{q}^n \int_{\mathcal{M}(r, n)} [\mathcal{M}(r, n)]_T,$$

where $[\mathcal{M}(r, n)]_T$ is the equivariant fundamental class of $H_T^*(\mathcal{M}(r, n))$. *Nekrasov's conjecture* says that the limit of $\varepsilon_1 \varepsilon_2 \log(\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2})$ for $\varepsilon_1, \varepsilon_2 \rightarrow 0$ is exactly \mathcal{F}^{inst} . This is proved in [89] and, independently, in [87].

For the case of an adjoint hypermultiplet of mass m , one can give a similar definition of instanton partition function for a $\mathcal{N} = 2^*$ $SU(r)$ -gauge theory on \mathbb{R}^4 as the generating function

$$\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*, inst}(\varepsilon_1, \varepsilon_2, a_1, \dots, a_r, m; \mathbf{q}) := \sum_{n=0}^{\infty} \mathbf{q}^n \int_{\mathcal{M}(r, n)} E_m(\mathcal{T}_{\mathcal{M}(r, n)}),$$

where $\mathcal{T}_{\mathcal{M}(r, n)}$ is the tangent bundle to the moduli space $\mathcal{M}(r, n)$, and the class E_m is defined for a vector bundle V of rank d as

$$E_m(V) := \sum_{j=0}^d (c_j)_T(V) m^{d-j}.$$

Also in this case one can state a version of Nekrasov's conjecture. By using the so-called *natural bundle* one can define the instanton partition function for gauge theories on \mathbb{R}^4 with fundamental matter. The natural bundle is defined by using the *universal sheaf* of $\mathcal{M}(r, n)$.

Since the T -fixed locus of $\mathcal{M}(r, n)$ consists of a finite number of fixed points, described by Young diagrams, one can apply the localization theorem in equivariant cohomology and obtain a combinatorial expression for the partition functions defined before depending on the equivariant parameters and the formal variable \mathbf{q} [88, 24].

In [88] the author introduced also the *deformed* Nekrasov partition function, which can be seen as the generating function of the equivariant cohomology version of Donaldson invariants on \mathbb{C}^2 . It is defined as

$$\mathcal{Z}_{\mathbb{R}^4}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}) := \sum_{n \geq 0} \mathbf{q}^n \int_{\mathcal{M}(r, n)} \exp \left(\sum_{p \geq 1} \tau_p \mathrm{ch}_{p+1}(\tilde{\mathcal{E}}) / [\mathbb{C}^2] \right),$$

where $\tilde{\mathcal{E}}$ is the universal sheaf of $\mathcal{M}(r, n)$ and ch_{p+1} denotes the degree $p+1$ part of the Chern character, and $/$ is the *slant* product, defined formally by localization since \mathbb{C}^2 is not compact. Setting $\vec{\tau} = (0, \tau_1, 0, \dots)$ one obtains the *instanton part* of the deformed Nekrasov partition function, which is of particular importance: it factorizes as a product of the instanton

partition function and the *classical part* $\mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a})$ of the Nekrasov partition function (see for example [15, Section 3.1]). So by using framed sheaves we can include the classical and the instanton partition functions in a unique partition function.

In [5] Alday, Gaiotto and Tachikawa uncovered a relation between two-dimensional conformal field theories (CFT) and a certain class of $\mathcal{N} = 2$ four-dimensional supersymmetric $SU(2)$ quiver gauge theories. In particular, it was argued that the conformal blocks in the Liouville field theory coincide with the instanton parts of the Nekrasov partition function. Further, this relation was generalized [6, 109] to CFTs with affine and $\mathcal{W}(\mathfrak{gl}_r)$ -symmetry. It turned out that the extended $\mathcal{W}(\mathfrak{gl}_r)$ conformal symmetry is related to the instanton counting for the $SU(r)$ gauge group.

This conjecture implies the existence of certain structures on the equivariant cohomology of the moduli space $\mathcal{M}(r, n)$ of framed sheaves on \mathbb{CP}^2 .

CONJECTURE (AGT conjecture for $\mathcal{N} = 2$ pure $SU(r)$ -gauge theory on \mathbb{R}^4). *Let $\vec{a} = (a_1, \dots, a_r)$. Define the vector space*

$$\mathbb{H}_{\varepsilon_1, \varepsilon_2, \vec{a}}^{\mathbb{R}^4}(r) := \bigoplus_{n \geq 0} H_T^*(\mathcal{M}(r, n)) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]} \mathbb{C}(\varepsilon_1, \varepsilon_2, a_1, \dots, a_r) .$$

Then

- (1) *The direct sum $\mathbb{H}_{\varepsilon_1, \varepsilon_2, \vec{a}}^{\mathbb{R}^4}(r)$ can be decomposed as*

$$\mathbb{H}_{\varepsilon_1, \varepsilon_2, \vec{a}}^{\mathbb{R}^4}(r) \cong \mathbb{V}_{Fock} \otimes M_{\vec{\beta}(\vec{a})}(c),$$

where \mathbb{V}_{Fock} is a Fock space of an Heisenberg algebra \mathcal{H} and $M_{\vec{\beta}(\vec{a})}(c)$ is the Verma module associated with a $\mathcal{W}(\mathfrak{gl}_r)$ -algebra with central charge c and momenta $\vec{\beta}(\vec{a})$ depending on the equivariant parameters.

- (2) **(Pure case).** *The vector $G := \sum_{n \geq 0} [\mathcal{M}(r, n)]_T$, in the extended vector space $\widehat{\mathbb{H}}_{\varepsilon_1, \varepsilon_2, \vec{a}}^{\mathbb{R}^4}(r)$, is a Whittaker vector with respect to $\mathcal{H} \times \mathcal{W}(\mathfrak{gl}_r)$.*

Note that the norm of the \mathbf{q} -deformed version $\sum_{n \geq 0} \mathbf{q}^n [\mathcal{M}(r, n)]_T$ of the vector G is exactly the instanton part $\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}$ of Nekrasov's partition function. This conjecture was proved by Schiffmann and Vasserot [101], by using a degenerate version of the double affine Hecke algebras, and independently by Maulik and Okounkov [80] by using Yangians.

One can state versions of the AGT conjecture for gauge theories with masses. From a mathematical viewpoint, these become statements about vertex operators which act on $\mathbb{H}_{\varepsilon_1, \varepsilon_2, \vec{a}}^{\mathbb{R}^4}(r)$. To the best of our knowledge, in the arbitrary rank case there are no proofs of these conjectures.

Motivations. Another class of Riemannian 4-manifolds over which it is very interesting to study gauge theories is the class of so-called *ALE spaces*, which are *deformation* of resolutions of \mathbb{R}^4/Γ where $\Gamma \subset SU(2)$ is a finite group. In the following we shall consider only ALE spaces of type A_{k-1} for $k \geq 2$. There have been in the physics literature various attempts to generalize the AGT conjecture to ALE spaces. In what follows we first introduce ALE spaces of type A_{k-1} for $k \geq 2$ and then describe how the conjecture has been translated in this picture.

Let $\mu_k \subset \mathbb{C}^*$ be the group of k -th roots of unity. By fixing a primitive root of unity ω , one can identify μ_k with \mathbb{Z}_k . Define an action of μ_k on \mathbb{C}^2 by $\omega \cdot (x, y) = (\omega x, \omega^{-1} y)$. The quotient $\mathbb{C}^2/\mathbb{Z}_k$ is a normal affine toric complex surface. Let us denote by $\pi_k: X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ its minimal resolution. By the *McKay correspondence* the irreducible components of $\pi_k^{-1}(0)$, which are $k - 1$ torus-invariant smooth projective curves of genus zero, are in a one-to-one correspondence with the vertices of the Dynkin diagram of type A_{k-1} . Moreover, the intersection matrix of such curves is exactly minus the Cartan matrix of the Dynkin diagram of type A_{k-1} .

An *ALE space of type A_{k-1}* is a smooth complex surface Y diffeomorphic to X_k endowed with a Kähler metric which is *Asymptotically Locally Euclidean*, i.e., there exists a compact subset $K \subset Y$ and a diffeomorphism $Y \setminus K \rightarrow (\mathbb{C}^2 \setminus B_r(0))/\mathbb{Z}_k$ under which the metric is approximated by the standard Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_k$ (see [69] for more details). An ALE space can be realized as a GIT quotient and it depends on a *stability parameter* ξ . The set of stability parameters has a *chamber structure*. For stability parameters in the same chamber, the ALE spaces are isomorphic, while for stability parameters in different chamber, they are just diffeomorphic. In the following we denote by Y_ξ an ALE space associated to the stability parameter ξ . In [70] the authors introduced the so-called *tautological line bundles* which are associated with the irreducible representations of μ_k , and form a basis for the second cohomology group of Y_ξ .

In [70], Kronheimer and Nakajima construct the moduli spaces parameterizing $U(r)$ -instantons on ALE spaces. In their construction they need not only to fix the topological invariants of the instantons but also their *holonomy at infinity*. We briefly recall what this latter notion means. Let (E, A) a $U(r)$ -instanton, where E is a holomorphic vector bundle of rank r on an ALE spaces Y equipped with a connection A whose curvature F_A is anti-selfdual and square-integrable. By Uhlenbeck's removable singularities theorem, E is *asymptotically flat*, i.e., there exists a flat connection A_0 on $E|_{Y \setminus K}$ such that the connection A is approximated by A_0 on $Y \setminus K$. Flat connections on $Y \setminus K$ are classified by their holonomies which take values in the fundamental group of $Y \setminus K$, which is \mathbb{Z}_k , hence a flat connection A_0 corresponds to a representation $\rho: \mathbb{Z}_k \rightarrow U(r)$. By fixing the topological invariants and the flat connection at infinity, Kronheimer and Nakajima give a characterization of $U(r)$ -instantons on an ALE space Y_ξ with fixed topological data and flat connection at infinity by means of linear data defined on vector spaces $V = \bigoplus_{i=0}^{k-1} \mathbb{C}^{v_i}$ and $W = \bigoplus_{i=0}^{k-1} \mathbb{C}^{w_i}$, where $\vec{v} = (v_0, \dots, v_{k-1}), \vec{w} = (w_0, \dots, w_{k-1}) \in \mathbb{Z}_{>0}^{\oplus k}$. Roughly speaking V provides information about the topological data of E and W about ρ . In this way, they construct moduli spaces $\mathcal{M}_\xi(\vec{v}, \vec{w})$ parametrizing these objects, which are smooth quasi-projective varieties. We shall call them *Nakajima quiver varieties of type \hat{A}_{k-1}* (the dependence on the extended Dynkin diagram of type \hat{A}_{k-1} is due to the relation of the quiver varieties with the representations theory of quivers). Also $\mathcal{M}_\xi(\vec{v}, \vec{w})$ is obtained as a GIT quotient with stability parameter ξ . As before, for stability parameters in the same chamber, the quiver varieties are isomorphic, while for stability parameters in different chambers, they are just diffeomorphic. For a particular choice of the invariants \vec{v}, \vec{w} (namely, $\vec{v} = (1, \dots, 1)$ and $\vec{w} = (1, 0, \dots, 0)$) one obtain the ALE space Y_ξ .

As pointed out in [43], there exists a chamber C_{orb} in the space of parameters such that for $\xi_{orb} \in C_{orb}$, $\mathcal{M}_{\xi_{orb}}(\vec{v}, \vec{w})$ is isomorphic to the moduli space of \mathbb{Z}_k -equivariant framed sheaves on \mathbb{P}^2 . The T -action of $\mathcal{M}(r, n)$ described before restricts to $\mathcal{M}_{\xi_{orb}}(\vec{v}, \vec{w})$. The fixed points

locus of $\mathcal{M}_\xi(\vec{v}, \vec{w})$ consists of a finite number of isolated points, which are in one-to-one correspondence to k -colored Young diagrams. As before, one can define the instanton partition function for a $\mathcal{N} = 2$ $U(r)$ -gauge theory on Y_ξ by generalizing the definitions of the partition functions we gave before. Since the quiver varieties are T -equivariantly diffeomorphic and they are isomorphic only if the stability parameters are in the same chamber, the computations in [43] provided instanton partition functions for gauge theories without mass and with adjoint mass on any ALE space Y_ξ and instanton partition functions for gauge theories with fundamental masses on ALE space Y_ξ with $\xi \in C_{orb}$.

Note that the parameter $\bar{\xi}$ such that $Y_{\bar{\xi}}$ is isomorphic to X_k is *not* in the chamber C_{orb} . Furthermore, one expects to find a *blow-up formula* for the instanton partition functions on ALE spaces in terms of instanton partition functions on \mathbb{R}^4 depending on equivariant parameters weighted by the affine patches of X_k (a first example of blow-up formula is in [87] for gauge theories on the blowup of \mathbb{C}^2 at the origin). This factorizations are not evident in the partition functions computed in [43].

In this picture one can ask the following questions:

QUESTION 1. *Can we find a suitable compactification of X_k , where to develop a theory of framed sheaves that provides another geometrical approach to the study of gauge theories on ALE spaces of type A_{k-1} ?*

Suppose we have a positive answer to this question. The next step is:

QUESTION 2. *Does this new geometrical approach allow us to compute partition functions for $U(r)$ -gauge theories on X_k , obtaining expressions in which the factorizations (blowup formulae) are evident?*

Moreover, thinking about the AGT conjecture on \mathbb{R}^4 , one can ask

QUESTION 3. *Can we give a mathematically rigorous version, and a proof, at least for $U(1)$ -gauge theories on X_k , of an AGT-type relation?*

Any possible answer Question 1 has to take in account a strong constraint. Bando proved in [10] that given a compact Kähler manifold X obtained as a compactification of a noncompact Kähler manifold X_0 by adding a smooth divisor D with positive normal line bundle, *there is a one-to-one correspondence between holomorphic vector bundles on X trivial along D and holomorphic vector bundles on X_0 endowed with an Hermitian Yang-Mills metric with trivial holonomy at infinity*. This means that the theory of framed vector bundles on smooth projective surfaces can describe only instantons with trivial holonomy at infinity. Thus one has to look for some more general *compactifications* of X_k .

A first indication for which direction to follow comes from Nakajima [85]: he suggested to take an *orbifold* compactification, in which the divisor D carries a \mathbb{Z}_k -action, in a way that the representations of \mathbb{Z}_k encode the holonomy at infinity of a framed vector bundle restricted to X_k . Other evidences are given by Bruzzo, Poghossian and Tanzini in [25], where they used framed sheaves on Hirzebruch surfaces \mathbb{F}_p , regarded as compactifications of the total space of the line bundles $\mathcal{O}_{\mathbb{P}^1}(-p)$, to compute the partition function of $\mathcal{N} = 4$ supersymmetric gauge theories on such spaces. They noted that their computations make sense also for *fractionary* classes $c_1(\mathcal{E}) \in \frac{1}{k}\mathbb{Z}C$, where C is the class of the section of $\mathbb{F}_p \rightarrow \mathbb{P}^1$ squaring to

$-p$. Although this does not make much sense, it suggested to the authors a conjecture: that their computations were actually taking place on a *stacky* compactification.

So a first goal in this thesis is to construct a 2-dimensional projective orbifold $\mathcal{X}_k = X_k \cup \mathcal{D}$, where \mathcal{D} is a smooth 1-dimensional closed substack with generic stabilizer μ_k , on which develop a theory of framed sheaves. This was motivated also by a work in progress by Eyssidieux and Sala [38], in which they are providing a correspondence between vector bundles on a 2-dimensional projective orbifold $\mathcal{X} = X_0 \cup \mathcal{D}$, isomorphic along \mathcal{D} to a fixed vector bundle \mathcal{F} , and holomorphic vector bundles on the Kähler surface X_0 , endowed with Hermitian Yang-Mills metrics with holonomy at infinity given by a fixed flat connection on \mathcal{F} .

For $U(1)$ -gauge theories we do not need this result, as Kuznetsov ([71]) proved that the *Hilbert scheme of points* $\text{Hilb}^n(X_k)$ is isomorphic to a Nakajima quiver variety of type \hat{A}_{k-1} with suitable dimensional vectors. In the thesis we shall prove that rank one framed sheaves on \mathcal{X}_k are equivalent to zero-dimensional subschemes of X_k .

Presentation of the results. This thesis represents part of a project in which we use framed sheaves on a *stacky compactification* of X_k to study gauge theories on X_k . In particular, the results we present here come from two joint works in progress, one with U. Bruzzo, F. Sala and R. Szabo [22] and the second with F. Sala and R. Szabo [96].

In Chapter 4 and the first part of Chapter 5 we answer Question 1.

Let $\bar{X}_k = X_k \cup D_\infty$ be the normal toric compactification of X_k with two singular points with the same type of singularity, obtained adding a divisor $D_\infty \cong \mathbb{P}^1$. For $k = 2$ this is actually smooth and coincide with the second Hirzebruch surface \mathbb{F}_2 . In general, it is a normal toric surface.

THEOREM 1. *There exists a 2-dimensional projective toric orbifold \mathcal{X}_k with the following properties.*

- (1) \mathcal{X}_k has \bar{X}_k as coarse moduli space, and the restriction of the coarse moduli space morphism

$$\pi|_{\pi^{-1}(X_k)}: \pi^{-1}(X_k) \subset \mathcal{X}_k \rightarrow X_k \subset \bar{X}_k$$

is an isomorphism.

- (2) The divisor $\mathcal{D}_\infty := \mathcal{X}_k \setminus X_k$ is an essentially trivial μ_k -gerbe over $\tilde{\mathcal{D}}_\infty$, where $\tilde{\mathcal{D}}_\infty := (\pi_k^{\text{can}})^{-1}(D_\infty)_{\text{red}}$ is a one dimensional, torus invariant, closed substack of the canonical stack $\pi_k^{\text{can}}: \mathcal{X}_k^{\text{can}} \rightarrow \bar{X}_k$ of \bar{X}_k .
- (3) The Picard group $\text{Pic}(\mathcal{X}_k)$ is a free abelian group of rank k . A basis is given by the line bundles \mathcal{R}_i , for $i = 1, \dots, k-1$ on \mathcal{X}_k , whose restrictions to X_k are the tautological line bundles, together with $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$.

\mathcal{X}_k is obtained by k -th root construction

$$\mathcal{X}_k = \sqrt[k]{\tilde{\mathcal{D}}_\infty / \mathcal{X}_k^{\text{can}}}$$

along $\tilde{\mathcal{D}}_\infty$ on the canonical stack $\mathcal{X}_k^{\text{can}}$. Consequently, the divisor \mathcal{D}_∞ is obtained as the k -root construction

$$\mathcal{D}_\infty = \sqrt[k]{\mathcal{O}_{\mathcal{X}_k}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty} / \tilde{\mathcal{D}}_\infty}.$$

The line bundles \mathcal{R}_i are defined as follows. Let

$$\mathcal{D}_i := \pi^{-1}(D_i)_{red}$$

for $i = 1, \dots, k-1$ be the divisors in \mathcal{X}_k corresponding to the exceptional divisors D_i of the resolution of singularities $X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$. Their intersection product is given by minus the Cartan matrix of type A_{k-1} . Define the dual classes

$$\omega_i := - \sum_{j=1}^{k-1} (C^{-1})_{i,j} [\mathcal{D}_j] .$$

In Lemma 4.22 we prove that they are integer classes, and in Proposition 4.24 we show that their associated line bundles $\mathcal{R}_i := \mathcal{O}_{\mathcal{X}_k}(\omega_i)$, together with the line bundle $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$, form a basis of $\text{Pic}(\mathcal{X}_k)$.

COROLLARY 2. *The Picard group $\text{Pic}(\mathcal{D}_\infty)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_k$, and the restriction to \mathcal{D}_∞ of the tautological line bundles \mathcal{R}_i gives the torsion part $\text{Pic}(\mathcal{D}_\infty)_{tor} \cong \mathbb{Z}_k$.*

In order to construct moduli spaces of framed sheaves which will be relevant from the gauge theoretic point of view, we need to choose a suitable locally free *framing sheaf*. Since the tautological line bundles introduced by Kronheimer are associated with the irreducible representation of μ_k , and the line bundles \mathcal{R}_i coincide with them on X_k , we choose the following. Fix $s \in \mathbb{Z}$. For $i = 0, \dots, k-1$ define the line bundles

$$\mathcal{O}_{\mathcal{D}_\infty}(s, i) := \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}^{\otimes s} \otimes \mathcal{R}_i|_{\mathcal{D}_\infty} .$$

Fix in addition $\vec{w} := (w_0, \dots, w_{k-1}) \in \mathbb{N}^k$ and define the locally free sheaf

$$\mathcal{F}_\infty^{s, \vec{w}} := \oplus_{i=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(s, i)^{\oplus w_i} .$$

In [38] the authors will show that the locally free sheaf $\mathcal{F}_\infty^{0, \vec{w}}$ carries a natural flat connection associated to the representation $\oplus_{i=0}^{k-1} \rho_i^{\oplus w_i}$, where ρ_i is the i -th irreducible representation of μ_k .

By Theorem 3.50 there exists a quasi-projective scheme $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ which is a fine moduli space for $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaves $(\mathcal{E}, \phi_{\mathcal{E}})$ on \mathcal{X}_k of fixed rank r , first Chern class $c_1(\mathcal{E}) = \sum_i u_i \omega_i$ and determinant $\Delta(\mathcal{E}) = \Delta$. In Theorem 5.9 we prove the following result.

THEOREM 3. *$\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ is a smooth quasi-projective variety of dimension*

$$\dim_{\mathbb{C}}(\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})) = 2r\Delta - \sum_{j=1}^{k-1} (C^{-1})_{j,j} \vec{w}(0) \cdot \vec{w}(j) ,$$

where the $\vec{w}(j)$'s are the vectors $(w_j, \dots, w_{k-1}, w_1, \dots, w_{j-1})$ and C is the Cartan matrix of type A_{k-1} .

When $r = 1$, we have a much nicer description of the moduli space $\mathcal{M}_{1, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$. In Proposition 5.10, it is shown that there is an isomorphism of fine moduli spaces

$$i: \text{Hilb}^\Delta(X_k) \xrightarrow{\sim} \mathcal{M}_{1, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$$

with the Hilbert scheme of Δ points of X_k .

In the second part of Chapter 5 we answer Question 2.

The torus T introduced before acts on $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ in a way similar to the one defined before for framed sheaves on \mathbb{P}^2 .

THEOREM 4. *The T -fixed points in $\mathcal{M}_{1,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ are in one-to-one correspondence with pairs (\vec{Y}, \vec{u}) , where*

- $\vec{Y} = (Y_1, \dots, Y_r)$ is a vector of r Young diagrams such that $\sum_{i=1}^r |Y_i| = n$,
- $\vec{u} = (u_1, \dots, u_r)$ such that $\sum_{\alpha=1}^r u_\alpha = \vec{u}$.

Moreover, we have the following constraint: set $\vec{v}_\alpha := C^{-1}u_\alpha$ for $\alpha = 1, \dots, r$. Then for $i = 0, \dots, k-1$ and $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^i w_j$ we have that

$$k(\vec{v}_\alpha)_i \equiv k - i \pmod{k},$$

and

$$\Delta = \sum_{\alpha=1}^r n_\alpha + \frac{r-1}{2r} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C\vec{v}_\alpha + \frac{1}{2r} \sum_{\alpha \neq \beta} \vec{v}_\alpha \cdot C\vec{v}_\beta \in \frac{1}{2rk} \mathbb{Z}.$$

We will denote a fixed point by its *combinatorial data* (\vec{Y}, \vec{u}) , or equivalently (\vec{Y}, \vec{v}) , where $\vec{v} = (v_1, \dots, v_r)$ defined above.

Introduce for $i = 1, \dots, k$

$$\begin{aligned} \varepsilon_1^{(i)} &= (k-i+1)\varepsilon_1 + (1-i)\varepsilon_2, \\ \varepsilon_2^{(i)} &= (i-k)\varepsilon_1 + i\varepsilon_2. \end{aligned}$$

Let $\vec{Y} = (Y_1, \dots, Y_r)$ be a vector of Young diagrams. Define for $i = 1, \dots, k$ the vectors $\vec{Y}^i := (Y_1^i, \dots, Y_r^i)$ and

$$(1) \quad \vec{a}^{(i)} := \vec{a} - (\vec{v})_i \varepsilon_1^{(i)} - (\vec{v})_{i-1} \varepsilon_2^{(i)},$$

where $(\vec{v})_l := ((v_1)_l, \dots, (v_r)_l)$ for $l = 1, \dots, k-1$ and $(\vec{v})_0 = (\vec{v})_k = 0$.

Following [86] we introduce the *deformed partition function* for supersymmetric gauge theories on X_k . Let $\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1}$ such that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$. Define

$$\begin{aligned} \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) &:= \sum_{\Delta \in \frac{1}{2rk} \mathbb{Z}} \mathbf{q}^{\Delta + \frac{1}{2r} \vec{v} \cdot C\vec{v}} \\ &\cdot \int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})} \exp \left(\sum_{p=0}^{\infty} \left(\sum_{i=1}^{k-1} t_p^{(i)} \left[\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i] \right]_p + \tau_p \left[\text{ch}_T(\tilde{\mathcal{E}})/[X_k] \right]_{p-1} \right) \right), \end{aligned}$$

where $\tilde{\mathcal{E}}$ is the *universal sheaf*, $\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i]$ denotes the *slant product* / between $\text{ch}_T(\tilde{\mathcal{E}})$ and $[\mathcal{D}_i]$ and the class $\text{ch}_T(\tilde{\mathcal{E}})/[X_k]$ is defined formally by localization (cf. [9, Section 3]) as

$$\text{ch}_T(\tilde{\mathcal{E}})/[X_k] := \sum_{i=1}^k \frac{1}{\text{Euler}(T_{p_i} X_k)} \iota_{\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})}^* \text{ch}_T(\tilde{\mathcal{E}});$$

here $\iota_{\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})}$ denotes the inclusion map of $\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ in $X_k \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$. Define also the *deformed partition function* for theories with

adjoint masses on X_k as

$$\begin{aligned} \mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) &:= \\ &= \sum_{\Delta \in \frac{1}{2rk}\mathbb{Z}} \mathbf{q}^{\Delta + \frac{1}{2r}\vec{v} \cdot C\vec{v}} \int_{\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})} E_m(\mathcal{TM}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})) \\ &\quad \cdot \exp \left(\sum_{p=0}^{\infty} \left(\sum_{i=1}^{k-1} t_p^{(i)} \left[\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i] \right]_p + \tau_p \left[\text{ch}_T(\tilde{\mathcal{E}})/[X_k] \right]_{p-1} \right) \right), \end{aligned}$$

where $\mathcal{TM}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})$ is the tangent bundle of $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})$. We call *deformed instanton part* of $\mathcal{Z}_{\vec{v}}$ the partition function $\mathcal{Z}_{\vec{v}}^{def-inst}$ obtained by setting $\vec{\tau} = (0, \tau_1, 0, \dots)$ and $\vec{t}^{(1)} = \dots = \vec{t}^{(k-1)} = 0$, and *instanton part* the partition function $\mathcal{Z}_{\vec{v}}^{inst}$ the one obtained setting also $\vec{\tau} = 0$. We define also the *instanton part* and the *deformed instanton part* of the deformed partition function for pure $U(r)$ -gauge theories on X_k as

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2, def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\xi}) &:= \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}}} \xi^{-\vec{v}} \mathcal{Z}_{\vec{v}}^{def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) \\ \mathcal{Z}_{ALE}^{\mathcal{N}=2, inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\xi}) &:= \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}}} \xi^{-\vec{v}} \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}), \end{aligned}$$

and we introduce the same partition functions for $U(r)$ -gauge theories with one adjoint hypermultiplet of mass m as

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2^*, def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\xi}) &:= \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}}} \xi^{-\vec{v}} \mathcal{Z}_{\vec{v}}^{*def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) \\ \mathcal{Z}_{ALE}^{\mathcal{N}=2^*, inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\xi}) &:= \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}}} \xi^{-\vec{v}} \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}), \end{aligned}$$

In Sections 5.3 and 5.4 we prove the following result.

THEOREM 5. *For the partition functions introduced above we have the following factorizations*

$$\mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) = \sum_{\vec{v}} \frac{\mathbf{q}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C \vec{v}_\alpha}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)})} \prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}; \mathbf{q}).$$

$$\mathcal{Z}_{\vec{v}}^{def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \tau_1) = \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1)^{\frac{1}{k}} \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}_{eff}, \tau_1),$$

where $\mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}_{eff}, \tau_1)$ is $\mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q})$ for $\tau_1 = 0$ (see Equation (85)).

$$\begin{aligned} \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}) &= \\ &= \sum_{\vec{v}} \mathbf{q}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C \vec{v}_\alpha} \frac{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)} + m)}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)})} \cdot \prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*, inst}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}, m; \mathbf{q}). \end{aligned}$$

$$\mathcal{Z}_{\vec{v}}^{*def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \tau_1) = \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1)^{\frac{1}{k}} \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}_{eff}, \tau_1) .$$

Here $\mathbf{q}_{eff} := \mathbf{q}e^{\tau_1}$, $\mathcal{Z}_{\mathbb{R}^4}^{cl}$, $\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2,inst}$, $\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*,inst}$ are respectively the classical part of the Nekrasov partition function, and the instanton part of the Nekrasov partition function for pure and adjoint masses $SU(r)$ -gauge theories on \mathbb{R}^4 , and the $\ell_{\alpha\beta}^{(j)}$ are the edge factors. Their explicit expression is computed in Appendix D and given in Formulae (68) and (69).

The expression for the edge factors in formulae (68) and (69) depends on the Cartan matrix. In [15], based on a conjectural splitting of the full partition function on X_k as a product of full partitions functions on the open affine substs U_i , the authors obtain an expression for the edge factors which depends just on the fan. At this stage a comparison of the two results does not appear to be easy, due to the different structures of the expressions. The following example however shows that for $k = 2$ the results coincide.

EXAMPLE 6. Focusing on the case $k = 2$, we have

$$\begin{aligned} \mathcal{Z}_v(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) &= \\ &= \sum_{\vec{v}} \frac{\mathbf{q}^{\sum_{\alpha=1}^r v_{\alpha}^2}}{\prod_{\alpha\beta} \ell_{\alpha\beta}^{(1)}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_{\alpha\beta}^{(1)})} \cdot \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2,inst}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a}^{(1)}; \mathbf{q}) \cdot \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*,inst}(\varepsilon_1 - \varepsilon_2, 2\varepsilon_2, \vec{a}^{(2)}; \mathbf{q}) , \end{aligned}$$

and

$$\begin{aligned} \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}) &= \\ &= \sum_{\vec{v}} \mathbf{q}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}} \frac{\prod_{\alpha\beta} \ell_{\alpha\beta}^{(1)}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_{\alpha\beta}^{(1)} + m)}{\prod_{\alpha\beta} \ell_{\alpha\beta}^{(1)}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_{\alpha\beta}^{(1)})} \\ &\quad \cdot \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*,inst}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a}^{(1)}, m; \mathbf{q}) \cdot \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*,inst}(\varepsilon_1 - \varepsilon_2, 2\varepsilon_2, \vec{a}^{(2)}, m; \mathbf{q}) . \end{aligned}$$

Introducing $a_{\alpha\beta}^{(1)} := a_{\beta} - a_{\alpha} - 2\{v_1\}\varepsilon_1^{(1)}$, we obtain

$$\ell_{\alpha\beta}^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\alpha\beta}^{(1)}) = \begin{cases} \prod_{i=0}^{\lfloor v_1 \rfloor - 1} \prod_{j=0}^{2\{v_1\} + 2i} \left((i + 2\{v_1\})\varepsilon_1^{(1)} + j\varepsilon_2^{(1)} + a_{\alpha\beta}^{(1)} \right)^{-1} & \text{for } v_1 \geq 0 , \\ \prod_{i=1}^{-\lfloor v_1 \rfloor} \prod_{j=1}^{2i-1-2\{v_1\}} \left((2\{v_1\} - i)\varepsilon_1^{(1)} - j\varepsilon_2^{(1)} + a_{\alpha\beta}^{(1)} \right) & \text{for } v_1 < 0 . \end{cases}$$

This agrees with [15, Formula 3.14] (see also the computations in [21, Section 4.2]). Note that we use different symbols than those in [15]: their \vec{k}_{α} are our \vec{v}_{α} , their $\vec{a}_{\alpha\beta}^{(i)}$ are the same of us. On the other hand, to fix the holonomy at infinity the authors use a vector \vec{I} of length r with components in $\{0, 1, \dots, k-1\}$ and then their \vec{k}_{α} satisfy an equation depending on \vec{I} . For us $\vec{I} = (0, \dots, 0, 1, \dots, 1, \dots, k-1, \dots, k-1)$, where i appears w_i -times for $i = 0, \dots, k-1$. \triangle

In the last Chapter 7 we answer question 3. Following [14], we consider the algebra $\mathcal{A}(1, k)$ obtained as a sum of an Heisenberg algebra \mathcal{H} and an affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_k$ of type \hat{A}_{k-1} , identifying their central elements. We prove the following.

THEOREM 7. Given $\gamma \in \mathfrak{Q}$, $n \in \mathbb{N}$, denote by $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$ the moduli space parameterizing isomorphism classes $[(\mathcal{E}, \phi_{\mathcal{E}})]$ of $(\mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{D}_{\infty}})$ -framed sheaves on \mathcal{X}_k of rank one, first Chern class given by γ and second chern number $\int_{\mathcal{X}_k} c_2(\mathcal{E}) = n$. Denote by $\mathbb{W}'_{\gamma, n}$ the localized

equivariant cohomology of $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$, and by $\mathbb{W}' := \oplus_{\gamma, n} \mathbb{W}'_{\gamma, n}$ the total localized equivariant cohomology. There exists an action of $\mathcal{A}(1, k)$ on \mathbb{W}' such that:

- (1) \mathbb{W}' is an irreducible, highest weight, level 1 $\mathcal{A}(1, k)$ -module, where level 1 means that the central element \mathfrak{c} acts as the identity. We will call this basic representation of $\mathcal{A}(1, k)$.
- (2) [Theorem 7.14 (**Pure case**)]. The *Gaiotto state*

$$G := \sum_{\gamma \in \mathfrak{Q}, n \in \mathbb{N}} [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \in \prod_{c \in \mathfrak{Q}, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n}$$

in the *completed* total localized equivariant cohomology $\hat{\mathbb{W}}' = \prod_{\gamma \in \mathfrak{Q}, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n}$ is a Whittaker vector of type $\chi: \mathcal{U}(\mathfrak{h}^+) \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$ with respect to this representation, where we have $\mathfrak{h} \cong \mathcal{H}^+ + \mathcal{H}_{\mathfrak{Q}}^+$, the sum of an Heisenberg algebra and a *lattice Heisenberg algebra* of type \mathfrak{Q} , and χ is defined by

$$\begin{aligned} \chi(h_i \otimes z^m) &= 0 \quad i = 1, \dots, k-1, m > 0, \\ \chi(\mathfrak{p}_m) &= \delta_{m,1} \sum_{i=1}^k \sqrt{\frac{\beta_i}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}}} (\beta_i^{-1} a_{i-1} - a_i) \quad m > 0. \end{aligned}$$

- (3) (**Adjoint multiplet case**). There exists a Carlsson-Okounkov type vertex operator

$$W(\mathcal{O}_{\mathcal{X}_k}(m), z) \in \text{End}(\mathbb{W}')[[z, z^{-1}]],$$

which can be written, in the standard generators of the Cartan subalgebra $\mathcal{H} + \mathcal{H}_{\mathfrak{Q}} \simeq \mathfrak{h} \subset \mathcal{A}(1, k)$, as

$$W_k(\mathcal{O}_{X_k}(m), z) = \exp \left(\sum_{i>0} \frac{(-1)^i z^i}{i} \sum_{j=1}^k \frac{m}{\varepsilon_2^{(i)}} \mathfrak{p}_{-i}^j \right) \exp \left(\sum_{i>0} \frac{(-1)^i z^{-i}}{i} \sum_{j=1}^k \frac{\varepsilon_1^{(i)} + \varepsilon_2^{(i)} - m}{\varepsilon_2^{(i)}} \mathfrak{p}_i^j \right).$$

such that

$$\text{str } \mathfrak{q}^{\mathcal{N}} \vec{\xi}^{\gamma} W(\mathcal{O}_{\mathcal{X}_k}(m), z) = \mathcal{Z}_{ALE}^{\mathcal{N}=2^*, \text{inst}}(\varepsilon_1, \varepsilon_2; \mathfrak{q}, \vec{\xi}).$$

where $\mathfrak{q}^{\mathcal{N}}$ is the box-counting operator (for $\gamma = 0$ is the usual box-counting operator), $\vec{\xi}^{\gamma}$ is the operator that counts $\gamma \in \mathfrak{Q}$, and $\mathcal{Z}_{ALE}^{\mathcal{N}=2^*, \text{inst}}$ is the instanton part of the deformed partition function for $\mathcal{N} = 2^*$ $U(1)$ -gauge theory on X_k .

The action of $\mathcal{A}(1, k)$ on \mathbb{W}' is obtained by *Frenkel-Kac construction*. We first construct analogs of Nakajima operators, obtaining a representation of a *rank k Heisenberg algebra* on the total localized equivariant cohomology of the Hilbert schemes of points $\text{Hilb}^n(X_k)$. By applying Frenkel-Kac construction to this representation, we get the action of $\mathcal{A}(1, k)$ on \mathbb{W}' .

Acknowledgements. The author wishes to thank Prof. Ugo Bruzzo and Dr. Francesco Sala for suggesting him these problems, for their willingness and constant support during his Ph.D. and for their help regarding this thesis. He would also like to thank Prof. Richard J. Szabo for his precious collaboration and for his work on the joint papers from which this thesis arises. Moreover he is grateful also to Prof. Alessandro Tanzini and Prof. Giulio Bonelli for the useful discussions about the physics behind these mathematical problems.

The author is grateful to the International School for Advanced Studies for funding his Ph.D., and to Heriot-Watt University for its hospitality and financial support during the visiting periods.

Conventions and notations

Our standard reference for the theory of stacks is [72]. We denote by k an algebraically closed field of characteristic zero. All schemes are defined over k , are Noetherian and of finite type, unless otherwise stated. A variety is a reduced separated scheme of finite type over k .

Let S be a generic base scheme of finite type over k . By *Deligne-Mumford S -stack* we mean a separated Noetherian Deligne-Mumford stack \mathcal{X} of finite type over S . We denote by $p: \mathcal{X} \rightarrow S$ the *structure morphism* of \mathcal{X} . When $S = \operatorname{Spec}(k)$, we do not mention the base scheme. For a Deligne-Mumford stack \mathcal{X} , we will write that x is a point of \mathcal{X} , or just $x \in \mathcal{X}$, meaning that x is an object in $\mathcal{X}(k)$. We denote by $\operatorname{Aut}(x)$ the automorphism group of the point x . We will say that a morphism between stacks is unique if it is unique up to a unique 2-arrow. An *orbifold* is a smooth Deligne-Mumford stack with generically trivial stabilizer.

The *inertia stack* $\mathcal{I}(\mathcal{X})$ of an algebraic stack \mathcal{X} is by definition the fibered product $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ with respect to the diagonal morphisms $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$. For a scheme T , an object in $\mathcal{I}(\mathcal{X})(T)$ consists of pairs (x, g) where x is an object of $\mathcal{X}(T)$ and $g: x \xrightarrow{\sim} x$ is an automorphism. A morphism $(x, g) \rightarrow (x', g')$ is a morphism $f: x \rightarrow x'$ in $\mathcal{X}(T)$ such that $f \circ g = g' \circ f$.

Let \mathcal{X} be a Deligne-Mumford S -stack. An *étale presentation* of \mathcal{X} is a pair (U, u) , where U is an S -scheme and $u: U \rightarrow \mathcal{X}$ is a *representable étale surjective* morphism (cf. [72, Definition 4.1]). A morphism between two étale presentations (U, u) and (V, v) of \mathcal{X} is a pair (φ, α) , where $\varphi: U \rightarrow V$ is a S -morphism and $\alpha: u \xrightarrow{\sim} v \circ \varphi$ is a 2-isomorphism. We call *étale groupoid* associated with the étale presentation $u: U \rightarrow \mathcal{X}$ the étale groupoid

$$V := U \times_{\mathcal{X}} U \rightrightarrows U.$$

If \mathbf{P} is a property of schemes which is local in the étale topology (for example regular, normal, reduced, Cohen-Macaulay, etc), the stack \mathcal{X} has the property \mathbf{P} if for one (and hence every) étale presentation $u: U \rightarrow \mathcal{X}$, the scheme U has the property \mathbf{P} .

A (quasi-)coherent sheaf \mathcal{E} on the stack \mathcal{X} is a collection of pairs $(\mathcal{E}_{U,u}, \theta_{\varphi,\alpha})$, where for any étale presentation $u: U \rightarrow \mathcal{X}$, $\mathcal{E}_{U,u}$ is a (quasi-)coherent sheaf on U , and for any morphism $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ between two étale presentations of \mathcal{X} , $\theta_{\varphi,\alpha}: \mathcal{E}_U \xrightarrow{\sim} \varphi^* \mathcal{E}_V$ is an isomorphism which satisfies a *cocycle condition* with respect to three étale presentations (cf. [72, Lemma 12.2.1], [107, Definition 7.18]). A locally free sheaf on \mathcal{X} is a coherent sheaf \mathcal{E} such that all representatives \mathcal{E}_U are locally free. We use indifferently the terms “locally free sheaf” and “vector bundle”. We denote by \mathbb{G}_m the sheaf of invertible sections in $\mathcal{O}_{\mathcal{X}}$.

If (\mathcal{X}, p) is a Deligne-Mumford S -stack, by [65, Corollary 1.3-(1)], there exist a separated algebraic space X and a morphism $\pi: \mathcal{X} \rightarrow X$ such that

- $\pi: \mathcal{X} \rightarrow X$ is proper and quasi-finite;
- if F is an algebraically closed field, $\mathcal{X}(\mathrm{Spec}(F))/\mathrm{Isom} \rightarrow X(\mathrm{Spec}(F))$ is a bijection;
- whenever $Y \rightarrow S$ is an algebraic space and $\mathcal{X} \rightarrow Y$ is a morphism, the morphism factors uniquely as $\mathcal{X} \rightarrow X \rightarrow Y$; more generally:
- whenever $S' \rightarrow S$ is a flat morphism of schemes, and whenever $Y \rightarrow S'$ is an algebraic space and $\mathcal{X} \times_S S' \rightarrow Y$ is a morphism, the morphism factors uniquely as $\mathcal{X} \times_S S' \rightarrow X \times_S S' \rightarrow Y$; in particular
- the natural morphism $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

We call the pair (X, π) a *coarse moduli space* of \mathcal{X} . If the coarse moduli space of \mathcal{X} is a scheme X , we call it a *coarse moduli scheme*. In this connection we recall some properties of Deligne-Mumford S -stacks that we shall use later:

- the functor $\pi_*: \mathrm{QCoh}(\mathcal{X}) \rightarrow \mathrm{QCoh}(X)$ is exact and maps coherent sheaves to coherent sheaves (cf. [4, Lemma 2.3.4]);
- $H^\bullet(\mathcal{X}, \mathcal{E}) \simeq H^\bullet(X, \pi_* \mathcal{E})$ for any quasi-coherent sheaf \mathcal{E} on \mathcal{X} (cf. [91, Lemma 1.10]);
- $\pi_* \mathcal{E}$ is an S -flat coherent sheaf on X whenever \mathcal{E} is an S -flat coherent sheaf on \mathcal{X} (cf. [91, Corollary 1.3-(3)]).

The projectivity of a scheme morphism is understood in the sense of Grothendieck, i.e., $f: X \rightarrow Y$ is projective if there exists a coherent sheaf E on Y such that f factorizes as a closed immersion of X into $\mathbb{P}(E)$ followed by the structural morphism $\mathbb{P}(E) \rightarrow Y$.

We use the letters $\mathcal{E}, \mathcal{F}, \mathcal{G}, \dots$, for sheaves on a Deligne-Mumford S -stack, and the letters E, F, G, \dots , for sheaves on a scheme. For any coherent sheaf \mathcal{F} on a Deligne-Mumford S -stack \mathcal{X} we denote by \mathcal{F}^\vee its *dual* $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathcal{X}})$. We denote in the same way the dual of a coherent sheaf on a scheme. A projection morphism $T \times Y \rightarrow Y$ is written as p_Y or $p_{T \times Y, Y}$.

CHAPTER 1

Projective, root and toric stacks

In this chapter we introduce some algebro-geometric preliminaries. In particular, in Section 1.1 we summarize some elements of the theory of projective stacks and coherent sheaves on them. In Section 1.2 we give an idea of the so-called *root construction* and study its main properties. The rest of the Chapter is devoted to the study of toric stacks, their properties and their connections with root stacks.

1.1. Projective stacks

In this section we introduce projective stacks and collect some elements of the theory of coherent sheaves on them. Our main references are [68, 91]. To define projective stacks one needs the notion of *tameness* (cf. [91, Definition 1.1]), but as in characteristic zero separatedness implies tameness (cf. [3]) and our Deligne-Mumford stacks are separated, we do not need to introduce that notion.

1.1.1. Generating sheaves. The projectivity of a scheme is related to the existence of a *very ample* line bundle on it (cf. [51]). In the stacky case, one can give an equivalent notion of projectivity only for a particular class of stacks. It was proven in [93] that, under certain hypotheses, there exist locally free sheaves, called *generating sheaves*, which behave like “very ample line bundles”. In [36], another class of locally free sheaves which resemble (very) ample line bundles was introduced. It was proved in [93] that these two classes of locally free sheaves coincide. We shall use one or the other definition according to convenience.

Let \mathcal{X} be a Deligne-Mumford S -stack with coarse moduli space $\pi: \mathcal{X} \rightarrow X$.

DEFINITION 1.1. Let \mathcal{G} be a locally free sheaf on \mathcal{X} . We define

$$\begin{aligned} F_{\mathcal{G}}: \mathcal{E} \in \mathrm{QCoh}(\mathcal{X}) &\longmapsto \pi_*(\mathcal{E} \otimes \mathcal{G}^\vee) \in \mathrm{QCoh}(X); \\ G_{\mathcal{G}}: E \in \mathrm{QCoh}(X) &\longmapsto \pi^*E \otimes \mathcal{G} \in \mathrm{QCoh}(\mathcal{X}). \end{aligned}$$

■

REMARK 1.2. The functor $F_{\mathcal{G}}$ is exact since \mathcal{G}^\vee is locally free and the direct image functor π_* is exact. The functor $G_{\mathcal{G}}$ is exact when the morphism π is flat. This happens for instance if the stack is a flat gerbe over a scheme, i.e., a stack over a scheme Y which étale locally admits a section and such that any two local sections are locally 2-isomorphic, or in the case of root stacks over schemes (we give a brief introduction to the theory of root stacks in Section 1.2). △

DEFINITION 1.3. A locally free sheaf \mathcal{G} is said to be a *generator* for the quasi-coherent sheaf \mathcal{E} if the adjunction morphism (left adjoint to the identity $\mathrm{id}: \pi_*(\mathcal{E} \otimes \mathcal{G}^\vee) \rightarrow \pi_*(\mathcal{E} \otimes \mathcal{G}^\vee)$)

$$(2) \quad \theta_{\mathcal{G}}(\mathcal{E}): \pi^*\pi_*(\mathcal{E} \otimes \mathcal{G}^\vee) \otimes \mathcal{G} \rightarrow \mathcal{E}$$

is surjective. It is a *generating sheaf* for \mathcal{X} if it is a generator for every quasi-coherent sheaf on \mathcal{X} . \blacksquare

A generating sheaf can be considered as a very ample sheaf relatively to the morphism $\pi: \mathcal{X} \rightarrow X$. Indeed, the property expressed by (2) resembles a similar property for very ample line bundles ([52, Theorem 2.1.1 Chap. III]): if $f: Y \rightarrow Z$ is a proper morphism, $\mathcal{O}_Y(1)$ is a very ample line bundle on Y relative to f , and E is coherent sheaf on Y , there is a positive integer N such that the adjunction morphism $f^*f_*\mathcal{H}om(\mathcal{O}_Y(-n), E) \otimes \mathcal{O}_Y(-n) \rightarrow E$ is surjective for any integer $n \geq N$.

Let E be a quasi-coherent sheaf on X . Since \mathcal{G} is locally free,

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\pi^*E \otimes \mathcal{G}, \pi^*E \otimes \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\pi^*E, \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{G}, \pi^*E \otimes \mathcal{G})) .$$

Define the morphism $\varphi_{\mathcal{G}}(E)$ as the right adjoint to the identity $\mathrm{id}: \pi^*E \otimes \mathcal{G} \rightarrow \pi^*E \otimes \mathcal{G}$:

$$\varphi_{\mathcal{G}}(E): E \rightarrow \pi_*(\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{G}, \pi^*E \otimes \mathcal{G})) = F_{\mathcal{G}}(G_{\mathcal{G}}(E)) .$$

LEMMA 1.4 (Projection Formula). [93, Corollary 5.4] *Let \mathcal{F} be a quasi-coherent sheaf on \mathcal{X} and E a quasi-coherent sheaf on X . A projection formula holds:*

$$\pi_*(\pi^*(E) \otimes \mathcal{F}) \simeq E \otimes \pi_*\mathcal{F} .$$

Moreover, this is functorial in the sense that if $f: \mathcal{F} \rightarrow \mathcal{F}'$ is a morphism of quasi-coherent sheaves on \mathcal{X} and $g: E \rightarrow E'$ is a morphism of quasi-coherent sheaves on X , one has

$$\pi_*(\pi^*(g) \otimes f) = g \otimes \pi_*f .$$

PROOF. This projection formula is proved at the beginning of the proof of Corollary 5.4 in [93]. \square

According to this Lemma, $\varphi_{\mathcal{G}}(E)$ can be rewritten as

$$E \xrightarrow{\varphi_{\mathcal{G}}(E)} E \otimes \pi_*(\mathrm{End}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{G})) ,$$

and is the morphism given by tensoring a section by the identity endomorphism; in particular it is injective.

LEMMA 1.5. [91, Lemma 2.9] *Let \mathcal{E} be a quasi-coherent sheaf on \mathcal{X} and L a coherent sheaf on X . The compositions*

$$\begin{aligned} F_{\mathcal{G}}(\mathcal{E}) &\xrightarrow{\varphi_{\mathcal{G}}(F_{\mathcal{G}}(\mathcal{E}))} F_{\mathcal{G}} \circ G_{\mathcal{G}} \circ F_{\mathcal{G}}(\mathcal{E}) \xrightarrow{F_{\mathcal{G}}(\theta_{\mathcal{G}}(\mathcal{E}))} F_{\mathcal{G}}(\mathcal{E}) \\ G_{\mathcal{G}}(L) &\xrightarrow{G_{\mathcal{G}}(\varphi_{\mathcal{G}}(L))} G_{\mathcal{G}} \circ F_{\mathcal{G}} \circ G_{\mathcal{G}}(L) \xrightarrow{\theta_{\mathcal{G}}(G_{\mathcal{G}}(L))} G_{\mathcal{G}}(L) . \end{aligned}$$

are the identity endomorphisms.

Following [36] we introduce another definition of “ampleness” for sheaves on stacks.

DEFINITION 1.6. A locally free sheaf \mathcal{V} on \mathcal{X} is π -*ample* if for every geometric point of \mathcal{X} the natural representation of the stabilizer group at that point on the fiber of \mathcal{V} is faithful. A locally free sheaf \mathcal{G} on \mathcal{X} is π -*very ample* if for every geometric point of \mathcal{X} the natural representation of the stabilizer group at that point on the fiber of \mathcal{G} contains every irreducible representation. \blacksquare

The relation between these two notions is explained in [68, Section 5.2]. In particular, we have the following result.

PROPOSITION 1.7. [23, Proposition 2.7] *Let \mathcal{V} be a π -ample sheaf on \mathcal{X} and N the maximum between the numbers of conjugacy classes of any geometric stabilizer group of \mathcal{X} . Then, for any $r \geq N$, the locally free sheaf $\bigoplus_{i=1}^r \mathcal{V}^{\otimes i}$ is π -very ample.*

As shown in [93, Theorem 5.2], a locally free sheaf \mathcal{V} on \mathcal{X} is π -very ample if and only if it is a generating sheaf.

REMARK 1.8. Let $\varphi: \mathcal{Y} \rightarrow \mathcal{X}$ be a representable morphism of Deligne-Mumford S -stacks. By the universal property of the coarse moduli spaces, φ induces a morphism $\bar{\varphi}: Y \rightarrow X$ between the corresponding coarse moduli spaces together with a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\varphi} & \mathcal{X} \\ \pi_{\mathcal{Y}} \downarrow & & \downarrow \pi_{\mathcal{X}} \\ Y & \xrightarrow{\bar{\varphi}} & X \end{array} \quad .$$

By [72, Proposition 2.4.1.3], for any geometric point of \mathcal{Y} the morphism φ induces an injective map between the stabilizer groups at that point and at the corresponding image point. So if \mathcal{V} is a $\pi_{\mathcal{X}}$ -ample sheaf on \mathcal{X} , then $\varphi^*\mathcal{V}$ is a $\pi_{\mathcal{Y}}$ -ample sheaf on \mathcal{Y} . Denote by $N_{\mathcal{X}}$ (resp. $N_{\mathcal{Y}}$) the maximum of the numbers of conjugacy classes of any geometric stabilizer group of \mathcal{X} (resp. \mathcal{Y}). If $N_{\mathcal{X}} \geq N_{\mathcal{Y}}$ by Proposition 1.7 we get that $\bigoplus_{i=1}^r \varphi^*\mathcal{V}^{\otimes i}$ is $\pi_{\mathcal{Y}}$ -very ample for any $r \geq N_{\mathcal{X}}$. \triangle

DEFINITION 1.9. [36, Definition 2.9] Let \mathcal{X} be a stack of finite type over a base scheme S . We say \mathcal{X} is a *global S -quotient* if it is isomorphic to a stack of the form $[T/G]$, where T is an algebraic space of finite type over S and G is an S -flat group scheme which is a group subscheme (a locally closed subscheme which is a subgroup) of the general linear group scheme $\mathrm{GL}_{N,S}$ over S for some integer N . \blacksquare

THEOREM 1.10. [93, Section 5]

- (i) *A Deligne-Mumford S -stack \mathcal{X} which is a global S -quotient always has a generating sheaf \mathcal{G} .*
- (ii) *Under the same hypotheses of (i), let $\pi: \mathcal{X} \rightarrow X$ be the coarse moduli space of \mathcal{X} and $f: X' \rightarrow X$ a morphism of algebraic spaces. Then $p_{\mathcal{X} \times_X X', \mathcal{X}}^* \mathcal{G}$ is a generating sheaf for $\mathcal{X} \times_X X'$.*

Now we are ready to give the definition of projective stack.

DEFINITION 1.11. [68, Definition 5.5] A Deligne-Mumford stack \mathcal{X} is a *(quasi-)projective stack* if \mathcal{X} admits a (locally) closed embedding into a smooth proper Deligne-Mumford stack which has a projective coarse moduli scheme. \blacksquare

PROPOSITION 1.12. [68, Theorem 5.3] *Let \mathcal{X} be a Deligne-Mumford stack. The following statements are equivalent:*

- (i) *\mathcal{X} is (quasi-)projective.*

- (ii) \mathcal{X} has a (quasi-)projective coarse moduli scheme and has a generating sheaf.
- (iii) \mathcal{X} is a separated global quotient with a coarse moduli space which is a (quasi-)projective scheme.

DEFINITION 1.13. Let \mathcal{X} be a projective stack with coarse moduli scheme X . A *polarization* for \mathcal{X} is a pair $(\mathcal{G}, \mathcal{O}_X(1))$, where \mathcal{G} is a generating sheaf of \mathcal{X} and $\mathcal{O}_X(1)$ is an ample line bundle on X . ■

We give a relative version of the notion of projective stacks.

DEFINITION 1.14. Let $p: \mathcal{X} \rightarrow S$ be a Deligne-Mumford S -stack which is a global S -quotient with a coarse moduli scheme X such that p factorizes as $\pi: \mathcal{X} \rightarrow X$ followed by a projective morphism $\rho: X \rightarrow S$. We call $p: \mathcal{X} \rightarrow S$ a *family of projective stacks*. ■

REMARK 1.15. Let $p: \mathcal{X} = [T/G] \rightarrow S$ be a family of projective stacks. For any geometric point $s \in S$ we have the following cartesian diagram

$$\begin{array}{ccc}
 \mathcal{X}_s & \longrightarrow & \mathcal{X} \\
 \downarrow \pi_s & \square & \downarrow \pi \\
 X_s & \longrightarrow & X \\
 \downarrow \rho_s & \square & \downarrow \rho \\
 s & \longrightarrow & S
 \end{array}$$

with $\mathcal{X}_s = [T_s/G_s]$, where T_s and G_s are the fibers of T and G , respectively. Since the morphism ρ is projective, the fibers X_s are projective schemes. The property of being coarse moduli spaces is invariant under base change, so that each X_s is a coarse moduli scheme for \mathcal{X}_s , and each \mathcal{X}_s is a projective stack. △

By Theorem 1.10, a family of projective stacks $p: \mathcal{X} \rightarrow S$ has a generating sheaf \mathcal{G} and the fiber of \mathcal{G} at a geometric point $s \in S$ is a generating sheaf for \mathcal{X}_s . This justifies the following definition.

DEFINITION 1.16. Let $p: \mathcal{X} \rightarrow S$ be a family of projective stacks. A *relative polarization* of $p: \mathcal{X} \rightarrow S$ is a pair $(\mathcal{G}, \mathcal{O}_X(1))$ where \mathcal{G} is a generating sheaf for \mathcal{X} and $\mathcal{O}_X(1)$ is an ample line bundle relative to $\rho: X \rightarrow S$. ■

1.1.2. Coherent sheaves on projective stacks. In this section we briefly recall the theory of coherent sheaves on projective stacks from [91, Section 3.1]. In particular, we shall see that the functor $F_{\mathcal{G}}$ preserves the dimension and the pureness of coherent sheaves on projective stacks. Let us fix a projective stack \mathcal{X} of dimension d , with a coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$, and a polarization $(\mathcal{G}, \mathcal{O}_X(1))$ on it.

REMARK 1.17. By [68, Proposition 5.1], the stack \mathcal{X} is of the form $[T/G]$ with T a quasi-projective scheme and G a linear algebraic group acting on T . This implies that the

category of coherent sheaves on \mathcal{X} is equivalent to the category of coherent G -equivariant sheaves on T (cf. [72, Example 12.4.6] and [107, Example 7.21]). In the following, we shall use this correspondence freely. \triangle

DEFINITION 1.18. Let \mathcal{E} be a coherent sheaf on \mathcal{X} . The *support* $\text{supp}(\mathcal{E})$ of \mathcal{E} is the closed substack associated with the ideal $\mathcal{I} = \ker(\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}))$. The dimension $\dim(\mathcal{E})$ of \mathcal{E} is the dimension of its support. We say that \mathcal{E} is a *pure sheaf of dimension* $\dim(\mathcal{E})$ if for any nonzero subsheaf \mathcal{G} of \mathcal{E} the support of \mathcal{G} is pure of dimension $\dim(\mathcal{E})$. We say that \mathcal{E} is *torsion-free* if it is a pure sheaf of dimension d . \blacksquare

REMARK 1.19. Let $u: U \rightarrow \mathcal{X}$ be an étale presentation of \mathcal{X} . Let \mathcal{E} be a coherent sheaf on \mathcal{X} of dimension d . First note that $u^*\mathcal{E}$ is exactly the representative $\mathcal{E}_{U,u}$ of \mathcal{E} on U . As explained in [91, Remark 3.3], $\text{supp}(u^*\mathcal{E}) \rightarrow \text{supp}(\mathcal{E})$ is an étale presentation of $\text{supp}(\mathcal{E})$. Moreover, $\dim(\mathcal{E}) = \dim(u^*\mathcal{E})$ and \mathcal{E} is pure if and only if $u^*\mathcal{E}$ is pure. \triangle

As it was shown in [91, Section 3] (cf. also [58, Definition 1.1.4]), there exists a unique filtration, the so-called *torsion filtration*, of a coherent sheaf \mathcal{E}

$$0 \subseteq T_0(\mathcal{E}) \subseteq T_1(\mathcal{E}) \subseteq \cdots \subseteq T_{\dim(\mathcal{E})-1}(\mathcal{E}) \subseteq T_{\dim(\mathcal{E})}(\mathcal{E}) = \mathcal{E},$$

where $T_i(\mathcal{E})$ is the maximal subsheaf of \mathcal{E} of dimension $\leq i$. Note that $T_i(\mathcal{E})/T_{i-1}(\mathcal{E})$ is zero or pure of dimension i . In particular, \mathcal{E} is pure if and only if $T_{\dim(\mathcal{E})-1}(\mathcal{E}) = 0$.

DEFINITION 1.20. The *saturation* of a subsheaf $\mathcal{E}' \subset \mathcal{E}$ is the minimal subsheaf $\bar{\mathcal{E}}'$ of \mathcal{E} containing \mathcal{E}' such that $\mathcal{E}/\bar{\mathcal{E}}'$ is zero or pure of dimension $\dim(\mathcal{E})$. \blacksquare

Clearly, the saturation of \mathcal{E}' is the kernel of the surjection

$$\mathcal{E} \rightarrow \mathcal{E}/\mathcal{E}' \rightarrow \frac{\mathcal{E}/\mathcal{E}'}{T_{\dim(\mathcal{E})-1}(\mathcal{E}/\mathcal{E}')}.$$

LEMMA 1.21. [91, Lemma 3.4] *Let \mathcal{X} be a projective stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$. Let \mathcal{E} be a coherent sheaf on \mathcal{X} . Then we have*

- (i) $\pi(\text{Supp}(\mathcal{E})) = \pi(\text{Supp}(\mathcal{E} \otimes \mathcal{G}^\vee)) \supseteq \text{Supp}(F_{\mathcal{G}}(\mathcal{E}));$
- (ii) $F_{\mathcal{G}}(\mathcal{E})$ is zero if and only if \mathcal{E} is zero.

PROPOSITION 1.22. [23, proposition 2.22] *Let \mathcal{X} be a projective stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$. A coherent sheaf \mathcal{E} on \mathcal{X} and the sheaf $F_{\mathcal{G}}(\mathcal{E})$ on X have the same dimension. Moreover, \mathcal{E} is pure if and only if $F_{\mathcal{G}}(\mathcal{E})$ is pure.*

PROOF. Assume first that \mathcal{E} is pure. Then the necessary part is proved in [91, Proposition 3.6]. For the sufficient part, let us consider the short exact sequence

$$(3) \quad 0 \rightarrow T_{\dim(\mathcal{E})-1}(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0.$$

Since the functor $F_{\mathcal{G}}$ is exact, we obtain

$$0 \rightarrow F_{\mathcal{G}}(T_{\dim(\mathcal{E})-1}(\mathcal{E})) \rightarrow F_{\mathcal{G}}(\mathcal{E}) \rightarrow F_{\mathcal{G}}(\mathcal{Q}) \rightarrow 0.$$

By Lemma 1.21, $\text{Supp}(F_{\mathcal{G}}(T_{\dim(\mathcal{E})-1}(\mathcal{E}))) \subseteq \pi(\text{Supp}(T_{\dim(\mathcal{E})-1}(\mathcal{E})))$, and since π preserves the dimensions, $\dim F_{\mathcal{G}}(T_{\dim(\mathcal{E})-1}(\mathcal{E})) \leq \dim \mathcal{E} - 1$. As by hypothesis $F_{\mathcal{G}}(\mathcal{E})$ is pure of dimension $\dim \mathcal{E}$, we have $F_{\mathcal{G}}(T_{\dim(\mathcal{E})-1}(\mathcal{E})) = 0$ and therefore $T_{\dim(\mathcal{E})-1}(\mathcal{E}) = 0$ by Lemma 1.21.

If \mathcal{E} is not pure, to prove the assertion it is enough to use the short exact sequence (3) and a similar argument as before applied to \mathcal{E} and \mathcal{Q} . \square

For pure coherent sheaves on \mathcal{X} , the functor $F_{\mathcal{G}}$ preserves the supports.

COROLLARY 1.23. [91, Corollary 3.8] *Let \mathcal{E} be a pure coherent sheaf on \mathcal{X} . Then*

$$\mathrm{Supp}(F_{\mathcal{G}}(\mathcal{E})) = \pi(\mathrm{Supp}(\mathcal{E})).$$

Further, the functor $F_{\mathcal{G}}$ is compatible with torsion filtrations.

COROLLARY 1.24. [91, Corollary 3.7] *The functor $F_{\mathcal{G}}$ sends the torsion filtration $0 \subseteq T_0(\mathcal{E}) \subseteq \cdots \subseteq T_{\dim(\mathcal{E})}(\mathcal{E}) = \mathcal{E}$ of \mathcal{E} to the torsion filtration of $F_{\mathcal{G}}(\mathcal{E})$, that is, $F_{\mathcal{G}}(T_i(\mathcal{E})) = T_i(F_{\mathcal{G}}(\mathcal{E}))$ for $i = 0, \dots, \dim(\mathcal{E})$.*

EXAMPLE 1.25. Let \mathcal{X} be a smooth projective stack and $\pi: \mathcal{X} \rightarrow X$ its coarse moduli scheme. By [91, Lemma 6.9], any torsion-free sheaf \mathcal{E} on \mathcal{X} fits into an exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{\vee\vee} \rightarrow \mathcal{Q} \rightarrow 0.$$

Let $u: U \rightarrow \mathcal{X}$ be an étale presentation of \mathcal{X} . In particular, U is a regular scheme of dimension $\dim(\mathcal{X})$ and u is a flat morphism. By applying the functor u^* , we obtain an exact sequence

$$0 \rightarrow u^*\mathcal{E} \rightarrow u^*\mathcal{E}^{\vee\vee} \rightarrow u^*\mathcal{Q} \rightarrow 0.$$

Note that $u^*\mathcal{E}^{\vee\vee} \simeq (u^*\mathcal{E})^{\vee\vee}$ (cf. [79]). Moreover, $\mathrm{codim} \mathcal{Q} \geq 2$ and $u^*(\mathcal{E})^{\vee\vee}$ is locally free except on a closed subset of U of codimension at least 3 (cf. [55, Section 1]). If $\dim(\mathcal{X}) = 1$, we obtain $\mathcal{Q} = 0$ and $u^*\mathcal{E}^{\vee\vee}$ is locally free. Thus $\mathcal{E}^{\vee\vee}$ is locally free and $\mathcal{E} \simeq \mathcal{E}^{\vee\vee}$. Therefore any torsion-free sheaf on a smooth projective stack of dimension one is locally free. If $\dim(\mathcal{X}) = 2$, then \mathcal{Q} is a zero-dimensional sheaf and $\mathcal{E}^{\vee\vee}$ is locally free. Thus we obtain the analog of the usual characterization of torsion-free sheaves on smooth curves and surfaces (cf. [58, Example 1.1.16]). \triangle

1.1.3. Hilbert polynomial. We define a polynomial which will be the analog of the usual Hilbert polynomial for coherent sheaves on projective schemes. Let us fix a projective stack \mathcal{X} of dimension d , with coarse moduli space $\pi: \mathcal{X} \rightarrow X$, and a polarization $(\mathcal{G}, \mathcal{O}_X(1))$ on it. (This was called *modified Hilbert polynomial* in [91]).

DEFINITION 1.26. The *Hilbert polynomial* of a coherent sheaf \mathcal{E} on \mathcal{X} is

$$P_{\mathcal{G}}(\mathcal{E}, n) := \chi(\mathcal{X}, \mathcal{E} \otimes \mathcal{G}^{\vee} \otimes \pi^*\mathcal{O}_X(n)) = \chi(X, F_{\mathcal{G}}(\mathcal{E}) \otimes \mathcal{O}_X(n)) = P(F_{\mathcal{G}}(\mathcal{E}), n).$$

■

By Proposition 1.22, $\dim F_{\mathcal{G}}(\mathcal{E}) = \dim(\mathcal{E})$. The function $n \mapsto P_{\mathcal{G}}(\mathcal{E}, n)$ is a polynomial with rational coefficients by [58, Lemma 1.2.1], and can be uniquely written in the form

$$P_{\mathcal{G}}(\mathcal{E}, n) = \sum_{i=0}^{\dim(\mathcal{E})} \alpha_{\mathcal{G}, i}(\mathcal{E}) \frac{n^i}{i!} \in \mathbb{Q}[n].$$

Moreover, the Hilbert polynomial is additive on short exact sequences since $F_{\mathcal{G}}$ is an exact functor (cf. Remark 1.2) and the Euler characteristic is additive on short exact sequences.

Let \mathcal{E} be a coherent sheaf on \mathcal{X} . We call *multiplicity* of \mathcal{E} the leading coefficient $\alpha_{\mathcal{G}, \dim(\mathcal{E})}(\mathcal{E})$ of its Hilbert polynomial. The *reduced Hilbert polynomial* of \mathcal{E} is

$$p_{\mathcal{G}}(\mathcal{E}, n) := \frac{P_{\mathcal{G}}(\mathcal{E}, n)}{\alpha_{\mathcal{G}, \dim(\mathcal{E})}(\mathcal{E})}.$$

The *hat-slope* of \mathcal{E} is

$$\hat{\mu}_{\mathcal{G}}(\mathcal{E}) := \frac{\alpha_{\mathcal{G}, \dim(\mathcal{E})-1}(\mathcal{E})}{\alpha_{\mathcal{G}, \dim(\mathcal{E})}(\mathcal{E})}.$$

For a d -dimensional coherent sheaf \mathcal{E} , its \mathcal{G} -*rank* is

$$\mathrm{rk}_{\mathcal{G}}(\mathcal{E}) := \frac{\alpha_{\mathcal{G}, d}(\mathcal{E})}{\alpha_d(\mathcal{O}_X)},$$

where $\alpha_d(\mathcal{O}_X)$ is the leading coefficient of the Hilbert polynomial of \mathcal{O}_X .

REMARK 1.27. Let \mathcal{E} be a coherent sheaf of dimension d . Let \mathcal{E}' be a d -dimensional coherent subsheaf of \mathcal{E} and $\bar{\mathcal{E}}'$ its saturation. Then $\mathrm{rk}_{\mathcal{G}}(\bar{\mathcal{E}}') = \mathrm{rk}_{\mathcal{G}}(\mathcal{E}')$ and $\hat{\mu}_{\mathcal{G}}(\bar{\mathcal{E}}') \geq \hat{\mu}_{\mathcal{G}}(\mathcal{E}')$. \triangle

1.1.3.1. *Smooth case.* If \mathcal{X} is smooth one can give another definition of rank of a coherent sheaf. Let \mathcal{E} be a d -dimensional coherent sheaf. The *rank* of \mathcal{E} is

$$(4) \quad \mathrm{rk}(\mathcal{E}) = \frac{1}{\alpha_d(\mathcal{O}_X)} \int_{\mathcal{X}}^{\mathrm{et}} \mathrm{ch}^{\mathrm{et}}(\mathcal{E}) [\pi^* c_1^{\mathrm{et}}(\mathcal{O}_X(1))]^d,$$

where $\mathrm{ch}^{\mathrm{et}}(\mathcal{E})$ is the *étale* Chern character of \mathcal{E} and $\int_{\mathcal{X}}^{\mathrm{et}}$ denotes the pushforward $p_*: H_{\mathrm{et}}^{\bullet}(\mathcal{X}) \rightarrow H_{\mathrm{et}}^{\bullet}(\mathrm{Spec}(k)) \simeq \mathbb{Q}$ of the morphism $p: \mathcal{X} \rightarrow \mathrm{Spec}(k)$, which is proper since \mathcal{X} is projective. (For a more detailed introduction of the étale cohomology of a Deligne-Mumford stack, we refer to [19], Appendix C.)

The *degree* of \mathcal{E} is

$$\deg_{\mathcal{G}}(\mathcal{E}) := \alpha_{\mathcal{G}, d-1}(\mathcal{E}) - \mathrm{rk}(\mathcal{E}) \alpha_{\mathcal{G}, d-1}(\mathcal{O}_{\mathcal{X}}),$$

and its *slope* is

$$\mu_{\mathcal{G}}(\mathcal{E}) := \frac{\deg_{\mathcal{G}}(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

In this case the (in)equalities in Remark 1.27 are still valid.

REMARK 1.28. Assume moreover that \mathcal{X} is an orbifold. Then the only codimension zero component of the inertia stack $\mathcal{I}(\mathcal{X})$ is \mathcal{X} (which is *associated* with the trivial stabilizer), so that, by the Töen-Riemann-Roch Theorem (see Appendix B), we get

$$\mathrm{rk}(\mathcal{E}) = \frac{\alpha_d(\mathcal{E})}{\alpha_d(\mathcal{O}_X)},$$

where $\alpha_d(\mathcal{E})$ is the leading coefficient of the Hilbert polynomial of $\pi_*(\mathcal{E})$. More details about the inertia stack and the Töen-Riemann-Roch Theorem will be given in Appendix B.

Let \mathcal{E} be a coherent sheaf on \mathcal{X} . Then $\mathrm{rk}(\mathcal{E})$ is the zero degree part $\mathrm{ch}_0^{\mathrm{et}}(\mathcal{E})$ of the étale Chern character of \mathcal{E} . This is a trivial check if \mathcal{E} is locally free. In general, we can note that by [68, Proposition 5.1], \mathcal{X} has the *resolution property*, i.e., any coherent sheaf on \mathcal{X} admits a surjective morphism from a locally free sheaf. Since \mathcal{X} is also smooth, the Grothendieck group of coherent sheaves on \mathcal{X} is isomorphic to the Grothendieck group of locally free sheaves on \mathcal{X} . Therefore $\mathrm{rk}(\mathcal{E}) = \mathrm{ch}_0^{\mathrm{et}}(\mathcal{E})$ for any coherent sheaf \mathcal{E} on \mathcal{X} . As a byproduct, we get $\mathrm{rk}_{\mathcal{G}}(\mathcal{E}) = \mathrm{rk}(\mathcal{G}) \mathrm{rk}(\mathcal{E})$. Moreover, we have the following relation between the hat-slope and the slope of \mathcal{E} , which is a generalization of the usual relation in the case of coherent sheaves on projective schemes (cf. [58, Section 1.6]):

$$(5) \quad \mu_{\mathcal{G}}(\mathcal{E}) = \mathrm{rk}(\mathcal{G}) \alpha_d(\mathcal{O}_X) \hat{\mu}_{\mathcal{G}}(\mathcal{E}) - \alpha_{\mathcal{G}, d-1}(\mathcal{O}_{\mathcal{X}}).$$

△

1.2. Root stacks

In this section we give a brief introduction to the theory of root stacks, as it was developed in [27] (see also [2]). These stacks are constructed, from a base stack, extending the generic stabilizer along a fixed divisor, which after this procedure becomes a gerbe.

1.2.1. Roots of a line bundle (with a global section). Let \mathcal{X} be an algebraic stack. We recall here a standard fact: there is an equivalence between the category of line bundles on \mathcal{X} and the category of morphisms $\mathcal{X} \rightarrow \mathcal{B}\mathbb{G}_m$, where the morphisms in the former category are taken to be isomorphisms of line bundles. Moreover, by [94, Example 5.13], there is an equivalence between the category of pairs (\mathcal{L}, s) , with \mathcal{L} line bundle on \mathcal{X} and $s \in \Gamma(\mathcal{X}, \mathcal{L})$, and the category of morphisms $\mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$, where \mathbb{G}_m acts on \mathbb{A}^1 by multiplication.

Let \mathcal{X} be an algebraic stack, \mathcal{L} a line bundle on \mathcal{X} , $s \in \Gamma(\mathcal{X}, \mathcal{L})$ a global section and k a positive integer. As explained above, the pair (\mathcal{L}, s) defines a morphism $\mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$. Let $\theta_k: [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism induced by

$$\begin{aligned} x \in \mathbb{A}^1 &\longmapsto x^k \in \mathbb{A}^1, \\ t \in \mathbb{G}_m &\longmapsto t^k \in \mathbb{G}_m. \end{aligned}$$

Under the previous correspondence, θ_k sends a pair (\mathcal{L}, s) to its k -th tensor power $(\mathcal{L}^{\otimes k}, s^{\otimes k})$.

DEFINITION 1.29. Let \mathcal{X} be an algebraic stack, \mathcal{L} a line bundle on \mathcal{X} , $s \in \Gamma(\mathcal{X}, \mathcal{L})$ and k a positive integer. We define the algebraic stack $\sqrt[k]{(\mathcal{L}, s)}/\mathcal{X}$ obtained from \mathcal{X} by k th root construction on (\mathcal{L}, s) to be the fibered product

$$\begin{array}{ccc} \sqrt[k]{(\mathcal{L}, s)}/\mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & \square & \downarrow \theta_k \\ \mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where the bottom morphism is the one corresponding to the pair (\mathcal{L}, s) . For brevity, we will call this type of stacks *root stacks*. ■

REMARK 1.30. When X is a scheme and L a line bundle with a global section s , we can describe explicitly the objects of $\sqrt[k]{(L, s)}/X(S)$ over a scheme S . These are quadruples $(S \xrightarrow{f} X, M, t, \phi)$, where f is a morphism of schemes, M is a line bundle on S , $t \in \Gamma(S, M)$, and $\phi: M^{\otimes k} \xrightarrow{\sim} f^*L$ is an isomorphism such that $\phi(t^{\otimes k}) = s$. One can define the arrows on $\sqrt[k]{(L, s)}/X(S)$ in a natural way. Given a morphism $\psi: S \rightarrow T$, the arrow

$$\psi^*: \sqrt[k]{(L, s)}/X(T) \rightarrow \sqrt[k]{(L, s)}/X(S)$$

is defined in the following way:

$$\psi^*: (T \xrightarrow{f} X, M, t, \phi) \longmapsto (S \xrightarrow{f \circ \psi} X, \psi^*M, \psi^*t, \psi^*\phi),$$

where

$$\psi^* \phi: \psi^* M^{\otimes k} \xrightarrow{\sim} \psi^* f^* L \xrightarrow{\sim} (f \circ \psi)^* L,$$

and the last isomorphism is canonically defined. \triangle

REMARK 1.31. As it is explained in [27, Example 2.4.2], if s is a nowhere vanishing section, then $\sqrt[k]{(\mathcal{L}, s)/\mathcal{X}} \simeq \mathcal{X}$. This shows that all the structure we add in $\sqrt[k]{(\mathcal{L}, s)/\mathcal{X}}$ is concentrated at the vanishing locus of s . \triangle

DEFINITION 1.32. Let \mathcal{X} be an algebraic stack, \mathcal{L} a line bundle on \mathcal{X} and k a positive integer. We define $\sqrt[k]{\mathcal{L}/\mathcal{X}}$ to be the algebraic stack obtained as the fibered product

$$\begin{array}{ccc} \sqrt[k]{\mathcal{L}/\mathcal{X}} & \longrightarrow & \mathcal{B}\mathbb{G}_m \\ \downarrow & \square & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{B}\mathbb{G}_m \end{array}$$

where $\mathcal{X} \rightarrow \mathcal{B}\mathbb{G}_m$ is determined by \mathcal{L} , and $\mathcal{B}\mathbb{G}_m \rightarrow \mathcal{B}\mathbb{G}_m$ is given by the map $\mathbb{G}_m \rightarrow \mathbb{G}_m$, $t \mapsto t^k$. \blacksquare

Let \mathcal{X} be an algebraic stack, \mathcal{L} a line bundle on \mathcal{X} . As it is described in [27, Example 2.4.3], $\sqrt[k]{\mathcal{L}/\mathcal{X}}$ is a closed substack of $\sqrt[k]{(\mathcal{L}, 0)/\mathcal{X}}$. In general, let \mathcal{D} be the vanishing locus of $s \in \Gamma(\mathcal{X}, \mathcal{L})$. We have a chain of inclusions of closed substacks

$$\sqrt[k]{\mathcal{L}|_{\mathcal{D}}/\mathcal{D}} \subset \sqrt[k]{(\mathcal{L}|_{\mathcal{D}}, 0)/\mathcal{D}} \subset \sqrt[k]{(\mathcal{L}, s)/\mathcal{X}}.$$

Moreover, $\sqrt[k]{\mathcal{L}|_{\mathcal{D}}/\mathcal{D}}$ is isomorphic to the reduced stack $\left(\sqrt[k]{(\mathcal{L}|_{\mathcal{D}}, 0)/\mathcal{D}}\right)_{red}$ associated with $\sqrt[k]{(\mathcal{L}|_{\mathcal{D}}, 0)/\mathcal{D}}$. Finally, by [27, Remark 2.2.3] there exists a cartesian diagram

$$\begin{array}{ccc} \sqrt[k]{(\mathcal{L}|_{\mathcal{D}}, 0)/\mathcal{D}} & \xrightarrow{j} & \sqrt[k]{(\mathcal{L}, s)/\mathcal{X}} \\ \downarrow & \square & \downarrow \\ \mathcal{D} & \xrightarrow{\iota} & \mathcal{X} \end{array}$$

and the commutative diagram

$$\begin{array}{ccc} \sqrt[k]{\mathcal{L}|_{\mathcal{D}}/\mathcal{D}} & \xrightarrow{i} & \sqrt[k]{(\mathcal{L}|_{\mathcal{D}}, 0)/\mathcal{D}} \\ \downarrow & \circlearrowleft & \downarrow \\ \mathcal{D} & \xrightarrow{id} & \mathcal{D} \end{array}$$

Locally, $\sqrt[k]{\mathcal{L}/\mathcal{X}}$ is a quotient of \mathcal{X} by a trivial action of μ_k , through this is not true globally. In general, $\sqrt[k]{\mathcal{L}/\mathcal{X}}$ is a μ_k -banded gerbe¹ over \mathcal{X} . Its cohomology class in the étale cohomology group $H^2(\mathcal{X}, \mu_k)$ is obtained from the class $[\mathcal{L}] \in H^1(\mathcal{X}, \mathbb{G}_m)$ via the boundary homomorphism $\delta: H^1(\mathcal{X}, \mathbb{G}_m) \rightarrow H^2(\mathcal{X}, \mu_k)$ obtained from the Kummer exact sequence

$$1 \longrightarrow \mu_k \longrightarrow \mathbb{G}_m \xrightarrow{(-)^k} \mathbb{G}_m \longrightarrow 1.$$

THEOREM 1.33 ([27]). *The projection $\sqrt[k]{(\mathcal{L}, s)/\mathcal{X}} \rightarrow \mathcal{X}$ is faithfully flat and quasi-compact. If X is a scheme and L a line bundle on it with global a section s , X is the coarse moduli scheme for both $\sqrt[k]{(L, s)}/X$ and $\sqrt[k]{L}/X$ with respect to the projections to X .*

1.2.2. Roots of an effective Cartier divisor. The correspondence above between pairs of a line bundle and a section over an algebraic stack \mathcal{X} , and morphism $\mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ can be generalized to n -tuples of line bundles, as stated in [27, Lemma 2.1.1]. Namely, there is an equivalence between the category of morphisms $\mathcal{X} \rightarrow [\mathbb{A}^n/\mathbb{G}_m^n]$ and the category of n -tuples $(\mathcal{L}_i, s_i)_{i=1}^n$, where each \mathcal{L}_i is a line bundle on \mathcal{X} , and $s_i \in \Gamma(\mathcal{X}, \mathcal{L}_i)$.

DEFINITION 1.34. Let \mathcal{X} be a smooth algebraic stack, $\vec{\mathcal{D}} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ be n effective Cartier divisors in \mathcal{X} , and $\vec{k} = (k_1, \dots, k_n)$ a vector of positive integers. Define the \vec{k} -root of \mathcal{X} with respect to $\vec{\mathcal{D}}$, $\sqrt[\vec{k}]{\vec{\mathcal{D}}/\mathcal{X}}$, to be the fibered product

$$\begin{array}{ccc} \sqrt[\vec{k}]{\vec{\mathcal{D}}/\mathcal{X}} & \longrightarrow & [\mathbb{A}^n/\mathbb{G}_m^n] \\ \downarrow & \square & \downarrow \theta_{\vec{k}} \\ \mathcal{X} & \longrightarrow & [\mathbb{A}^n/\mathbb{G}_m^n] \end{array}$$

where $\theta_{\vec{k}} := \theta_{k_1} \times \dots \times \theta_{k_n}: [\mathbb{A}^n/\mathbb{G}_m^n] \rightarrow [\mathbb{A}^n/\mathbb{G}_m^n]$, and $\mathcal{X} \rightarrow [\mathbb{A}^n/\mathbb{G}_m^n]$ is the morphism determined by $(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i), s_{\mathcal{D}_i})_{i=1}^n$. We denote by $s_{\mathcal{D}_i}$ the tautological section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$ which vanishes along \mathcal{D}_i . \blacksquare

The top arrow in the diagram above $\sqrt[\vec{k}]{\vec{\mathcal{D}}/\mathcal{X}} \rightarrow [\mathbb{A}^n/\mathbb{G}_m^n]$ corresponds to n effective divisors $(\tilde{\mathcal{D}}_1, \dots, \tilde{\mathcal{D}}_n)$, where each $\tilde{\mathcal{D}}_i$ is the reduced closed substack $\pi^{-1}(\mathcal{D}_i)_{red}$, and $\pi: \sqrt[\vec{k}]{\vec{\mathcal{D}}/\mathcal{X}} \rightarrow \mathcal{X}$ is the natural projection morphism. Moreover,

$$\mathcal{O}_{\sqrt[\vec{k}]{\vec{\mathcal{D}}/\mathcal{X}}}(\tilde{\mathcal{D}}_i)^{\otimes k_i} \simeq \pi^*(\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)).$$

Note also that by [27, Remark 2.2.5],

$$\sqrt[\vec{k}]{\vec{\mathcal{D}}/\mathcal{X}} \cong \sqrt[k_1]{\mathcal{D}_1/\mathcal{X}} \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \sqrt[k_n]{\mathcal{D}_n/\mathcal{X}}.$$

¹A gerbe $\mathcal{Y} \rightarrow \mathcal{X}$ is a μ_k -banded gerbe, or simply a μ_k -gerbe, if for every étale chart U of \mathcal{X} and every object $x \in \mathcal{Y}(U)$ there is an isomorphism $\alpha_x: \mu_{k|U} \rightarrow \text{Aut}_U(x)$ of sheaves of groups, such that the natural compatibility conditions are satisfied.

REMARK 1.35. As explained in [12, Section 2.1], since \mathcal{X} and the divisors \mathcal{D}_i are smooth and each \mathcal{D}_i has simple normal crossing, then $\sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}}$ is a smooth algebraic stack and $\tilde{\mathcal{D}}_i$ have simple normal crossing. Moreover, $\tilde{\mathcal{D}}_i$ is the root stack $\sqrt[k_i]{\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)|_{\mathcal{D}_i}/\mathcal{D}_i}$, hence it is a μ_{k_i} -banded gerbe over \mathcal{D}_i . Since the class $[\tilde{\mathcal{D}}_i]$ has trivial image in $H^2(\mathcal{D}_i, \mathbb{G}_m)$, the gerbe $\tilde{\mathcal{D}}_i$ is *essentially trivial* (cf. [77, Definition 2.3.4.1 and Lemma 2.3.4.2]). \triangle

1.2.3. Picard groups of root stacks. We conclude this section by giving a useful characterization of the Picard group of $\sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}}$: we have the following morphism of exact sequences of groups (cf. [27, Corollary 3.1.2] and [39, 1.3.b, diagram (1.4)])

$$(6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\cdot \vec{k}} & \mathbb{Z}^n & \longrightarrow & \prod_{i=1}^n \mathbb{Z}_{k_i} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Pic}(\mathcal{X}) & \xrightarrow{\pi^*} & \text{Pic}\left(\sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}}\right) & \xrightarrow{q} & \prod_{i=1}^n \mathbb{Z}_{k_i} \longrightarrow 0 \end{array}$$

Every line bundle $\mathcal{L} \in \text{Pic}\left(\sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}}\right)$ can be written in a unique way as $\mathcal{L} \simeq \pi^*(\mathcal{M}) \otimes \bigotimes_{i=1}^n \mathcal{O}(\tilde{\mathcal{D}}_i)^{\otimes m_i}$, where $\mathcal{M} \in \text{Pic}(\mathcal{X})$ and $0 \leq m_i < k_i$. Moreover, the m_i 's are unique and \mathcal{M} is unique up to isomorphism. The morphism q maps \mathcal{L} to $(m_i)_{i=1}^n$.

LEMMA 1.36. [27, Theorem 3.1.1] *Let \mathcal{X} be an algebraic stack, \mathcal{F} a coherent sheaf on \mathcal{X} . For any integer m , we have*

$$\pi_* \left(\pi^*(\mathcal{M}) \otimes \bigotimes_{i=1}^n \mathcal{O}_{\sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}}}(\tilde{\mathcal{D}}_i)^{\otimes m_i} \right) \simeq \mathcal{M} \otimes \bigotimes_{i=1}^n \mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)^{\otimes \lfloor m_i/k_i \rfloor}.$$

1.3. Toric varieties

Here we recall some results about toric varieties that can be found in [44] and [33]. The main construction in which we are interested in is the description of a toric variety as a global quotient, described by Cox in [32].

Consider a toric variety X , and let T be its torus. Denote by $M = T^\vee := \text{Hom}(T, \mathbb{C}^*)$ the character lattice and by $N := \text{Hom}(M, \mathbb{Z})$ the lattice of one-parameter subgroups. Then we know X corresponds to a *fan* $\Sigma \subset N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$.

DEFINITION 1.37. Let $\Sigma \subset N_{\mathbb{Q}}$ be a fan. A cone $\sigma \in \Sigma$ is said to be

- (a) *simplicial* if its minimal generators are linearly independent over \mathbb{Q} ,
- (b) *smooth* if its minimal generators form a part of a \mathbb{Z} -basis of N .

The fan Σ is *simplicial* (resp. *smooth*) if every $\sigma \in \Sigma$ is simplicial (resp. smooth). We say also that Σ is *complete* if its support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ is all $N_{\mathbb{Q}}$. \blacksquare

We now give a characterization of the properties of a toric variety corresponding to the properties of its fan defined above.

THEOREM 1.38. [33, Theorem 3.1.19] *Let X_Σ be the toric variety corresponding to the fan Σ . Then*

- (a) X_Σ is normal with only finite quotient singularities if and only if the fan Σ is simplicial;
- (b) X_Σ is smooth if and only if the fan Σ is smooth;
- (c) X_Σ is proper if and only if the fan Σ is complete.

From now on, we assume the fan Σ to be simplicial. Let $\rho_1, \dots, \rho_n \in \Sigma(1)$ be the rays (one-dimensional cones), and for each i denote by v_i the unique generator of $\rho_i \cap N$. Let D_i be the irreducible torus-invariant Weil divisor corresponding to the ray ρ_i , and denote by $\text{Div}^T(X)$ the free abelian groups of T -invariant Weil divisor. Then we can define a map $i : M \rightarrow \text{Div}^T(X)$ by sending a character $m \in M$ to $\sum_{i=1}^n m(v_i) \in \text{Div}^T(X)$. If we assume that the rays ρ_i span $N_{\mathbb{Q}}^2$, the map i is injective and fits into a short exact sequence of abelian groups

$$(7) \quad 0 \rightarrow M \xrightarrow{i} \text{Div}^T(X) \rightarrow \text{Cl}(X) \simeq A^1(X) \rightarrow 0,$$

where $\text{Cl}(X)$ is the class groups, i.e., the Chow group $A^1(X)$. For an abelian group A denote by G_A the diagonalizable group $G_A := \text{Hom}(A, \mathbb{C}^*)$. Then we have an induced short exact sequence of diagonalizable groups

$$(8) \quad 1 \rightarrow G_{\text{Cl}(X)} \rightarrow G_{\text{Div}^T(X)} \rightarrow T \rightarrow 1.$$

Define $Z_\Sigma \subset \mathbb{C}^n$ to be the $G_{\text{Div}^T(X)} = (\mathbb{C}^*)^n$ -invariant open subset defined by $Z_\Sigma := \bigcup_{\sigma \in \Sigma} Z_\sigma$, with $Z_\sigma := \{x \in \mathbb{C}^n \mid x_i \neq 0 \text{ if } \rho_i \notin \sigma\}$. Here $G_{\text{Div}^T(X)} = (\mathbb{C}^*)^n$ acts on $Z_\Sigma \subset \mathbb{C}^n$ via the natural action on each coordinate. The first morphism in the short exact sequence (8) induces an action of $G_{\text{Cl}(X)}$ on Z_Σ , which has finite stabilizers as the fan is assumed to be simplicial. Then by [32, Theorem 2.1] X is the geometric quotient $Z_\Sigma / G_{\text{Cl}(X)}$, with torus $T \simeq G_{\text{Div}^T(X)} / G_{\text{Cl}(X)} = (\mathbb{C}^*)^n / G_{\text{Cl}(X)}$. Moreover, for any $i = 1, \dots, n$, the T -invariant Weil divisor $D_i \subset X$ is the geometric quotient

$$(Z_\Sigma \cap \{x_i = 0\}) / G_{\text{Cl}(X)}.$$

If X is also smooth, the natural morphism $D_i \in \text{Div}^T(X) \mapsto \mathcal{O}_X(D_i) \in \text{Pic}(X)$ is surjective and has kernel M , i.e., it establishes a natural isomorphism $\text{Cl}(X) \simeq \text{Pic}(X)$, and so $G_{\text{Cl}(X)} \simeq \text{Hom}(\text{Pic}(X), \mathbb{C}^*)$.

1.4. Picard stacks and Deligne-Mumford tori

Here we aim to define the analog of *tori* in toric geometry. They are the so-called *Deligne-Mumford* tori, as defined by Fantechi, Mann and Nironi in [39]. For this we first define *Picard stacks*. Then we define Deligne-Mumford tori as Picard stacks associated with certain morphisms of finite abelian groups.

²This is not a strong assumption. Indeed, it is equivalent to assume that the toric variety X is not of the form $\tilde{X} \times \tilde{T}$ where \tilde{T} is a torus. If we are in this case, then the assumption is true for \tilde{X} .

1.4.1. Picard stacks. In this section we introduce *Picard stacks* and morphisms between them as defined in [8, Exp. XVIII].

DEFINITION 1.39. A *Picard stack* \mathcal{G} over a base scheme S is a stack together with the following data:

- a stack morphism $m: \mathcal{G} \times_S \mathcal{G} \rightarrow \mathcal{G}$, denoted also by $m(g_1, g_2) = g_1 \cdot g_2$, also called *multiplication*;
- a 2-arrow θ called *associativity*: $\theta_{g_1, g_2, g_3}: (g_1 \cdot g_2) \cdot g_3 \Rightarrow g_1 \cdot (g_2 \cdot g_3)$;
- a 2-arrow τ called *commutativity*: $\tau_{g_1, g_2}: g_1 \cdot g_2 \Rightarrow g_2 \cdot g_1$.

This data must satisfy some compatibility conditions:

- (1) Given any chart U and any object $g \in \mathcal{G}(U)$ the morphism of stacks $m_g: \mathcal{G} \rightarrow \mathcal{G}$, which multiplies every object by g and any arrow by id_g , is an isomorphism.
- (2) (*Pentagon relation*) For any chart U and any choice of objects $g_1, \dots, g_4 \in \mathcal{G}(U)$,

$$(\text{id}_{g_1} \cdot \theta_{g_2, g_3, g_4}) \circ \theta_{g_1, g_2 \cdot g_3, g_4} \circ (\theta_{g_1, g_2, g_3} \cdot \text{id}_{g_4}) = \theta_{g_1, g_2, g_3 \cdot g_4} \circ \theta_{g_1 \cdot g_2, g_3, g_4}.$$

- (3) For any chart U and any object $g \in \mathcal{G}(U)$, one has $\tau_{g, g} = \text{id}_g$.
- (4) For any chart U and any choice of objects $g_1, g_2 \in \mathcal{G}(U)$, one has $\tau_{g_1, g_2} \circ \tau_{g_2, g_1} = \text{id}_{g_2 \cdot g_1}$.
- (5) (*Hexagon relation*) For any chart U and any choice of objects $g_1, g_2, g_3 \in \mathcal{G}(U)$, one has

$$\theta_{g_1, g_2, g_3} \circ \tau_{g_3, g_1, g_2} \circ \theta_{g_3, g_1, g_2} = (\text{id}_{g_1} \cdot \tau_{g_2, g_3}) \circ \theta_{g_1, g_3, g_2} \circ (\tau_{g_3, g_1} \cdot \text{id}_{g_2}).$$

■

REMARK 1.40. We can understand some of the previous relations by thinking of them as the usual group law: the pentagon relation is the analog of an associativity condition, the condition (3) means that every object commutes with himself, while the hexagon relation is the compatibility between associativity and commutativity 2-arrows. By [8, 1.4.4, Exp. XVIII] the previous definition imply the existence of a neutral element, i.e., a pair (e, ϵ) where $e: S \rightarrow \mathcal{G}$ is a section and $\epsilon: e \cdot e \Rightarrow e$. \triangle

DEFINITION 1.41. Let \mathcal{G}, \mathcal{H} be two Picard S -stacks. A *morphism of Picard S -stacks* $F: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of S -stacks together with a 2-arrow $\phi_{g_1, g_2}: F(g_1 \cdot g_2) \Rightarrow F(g_1) \cdot F(g_2)$ for any pair g_1, g_2 of objects of \mathcal{G} such that:

- Given any chart U and every pair of objects $g_1, g_2 \in \mathcal{G}(U)$

$$(\tau_{\mathcal{H}})_{F(g_1), F(g_2)} \circ \phi_{g_1, g_2} = \phi_{g_2, g_1} \circ F((\tau_{\mathcal{G}})_{g_1, g_2}).$$

- Given any chart U and every triple of objects $g_1, g_2, g_3 \in \mathcal{G}(U)$

$$\begin{aligned} \phi_{g_1, g_2 \cdot g_3} \circ (\text{id}_{F(g_1)} \cdot \phi_{g_2, g_3}) \circ F((\theta_{\mathcal{G}})_{g_1, g_2, g_3}) &= \\ &= (\theta_{\mathcal{H}})_{F(g_1), F(g_2), F(g_3)} \circ (\phi_{g_1, g_2} \cdot \text{id}_{F(g_3)}) \circ \phi_{g_1 \cdot g_2, g_3}. \end{aligned}$$

■

REMARK 1.42. Note that if $(e_{\mathcal{G}}, \epsilon_{\mathcal{G}})$ is a neutral element for \mathcal{G} , the pair $(F(e_{\mathcal{G}}), F(\epsilon_{\mathcal{G}}) \circ \phi_{e_{\mathcal{G}}, e_{\mathcal{G}}}^{-1})$ is a neutral element for \mathcal{H} . \triangle

The Picard stacks over S form a category with Picard stacks as objects and equivalence classes of morphism of Picard stacks as morphism.

REMARK 1.43. With a given a complex $G^\bullet := [G^{-1} \rightarrow G^0]$ of sheaves of abelian groups over S , one can associate a Picard stack \mathcal{G} [39, Remark 1.12]. If G^\bullet is a complex of diagonalizable groups, the associated Picard stack is the quotient stack $[G^{-1}/G^0]$. Denote by $D^{[-1,0]}(S, \mathbb{Z})$ the derived category of length 1 complexes of sheaves of abelian groups over S . Then associating a Picard stack gives a functor from $D^{[-1,0]}(S, \mathbb{Z})$ to the category of Picard stacks. \triangle

PROPOSITION 1.44. [8, Proposition 1.4.15] *The functor from $D^{[-1,0]}(S, \mathbb{Z})$ to the category of Picard stacks that associates with a length 1 complex of sheaves of abelian groups a Picard stack induces an equivalence of categories.*

In particular given any sheaf G of abelian groups over the base scheme S , the gerbe $\mathcal{B}G$, which is the quotient stack $[S/G]$ is naturally a Picard stack.

Now we will introduce the notion of an action of a Picard stack on a stack. The definition, given in [39], is a generalization of the definition of action of a group scheme on a stack given by Romagny in [100].

DEFINITION 1.45. ([39, Definition 1.14]) Let \mathcal{G} be a Picard stack and \mathcal{X} a stack. Denote by e and ϵ the neutral section and the corresponding 2 arrow. An *action* of \mathcal{G} on \mathcal{X} is given by:

- a morphism of stacks $a: \mathcal{G} \times_S \mathcal{X} \rightarrow \mathcal{X}$, denoted by $a(g, x) = g \cdot x$;
- a 2-arrow $\eta_x: e \cdot x \Rightarrow x$ for any object x of \mathcal{X} ;
- a 2-arrow $\sigma_{g_1, g_2, x}: (g_1 \cdot g_2) \cdot x \Rightarrow g_1 \cdot (g_2 \cdot x)$ for any two objects g_1, g_2 of \mathcal{G} and any object x of \mathcal{X} , called *associativity*.

These data are subject to the conditions:

- (*Pentagon relation*) Given any chart U , any three objects $g_1, g_2, g_3 \in \mathcal{G}(U)$ and any object $x \in \mathcal{X}(U)$, one has

$$(\text{id}_{g_1} \cdot \sigma_{g_2, g_3, x}) \circ \sigma_{g_1, g_2, g_3, x} \circ (\theta_{g_1, g_2, g_3} \cdot \text{id}_x) = \sigma_{g_1, g_2, g_3, x} \circ \sigma_{g_1 \cdot g_2, g_3, x}.$$

- Given any chart U and any object $x \in \mathcal{X}(U)$, one has

$$(\text{id}_e \cdot \eta_x) \circ \sigma_{e, e, x} = (\epsilon \cdot \text{id}_x).$$

■

Note that the multiplication map m of a Picard stack \mathcal{G} induces a natural action of the stack on itself.

1.4.2. Deligne-Mumford tori. Now we will define the objects that will play, for the toric stacks, the role of the tori for the toric varieties, namely the Deligne-Mumford tori. These are particular type of Picard stack associated with complexes of diagonalizable groups. We follow [39, Section 2].

Consider a morphism $\phi: A^0 \rightarrow A^1$ of finitely generated abelian groups, such that $\ker(\phi)$ is free. Then by [39, Lemma 2.1], the complex $[A^0 \rightarrow A^1]$ is isomorphic, in the derived

category of length 1 complexes of finitely generated abelian groups, to the complex $[\ker \phi \xrightarrow{0} \operatorname{coker} \phi]$. By applying the functor $\operatorname{Hom}(\cdot, \mathbb{C}^*)$ we obtain a length 1 complex of diagonalizable groups $[G_{A^1} \rightarrow G_{A^0}]$. By Remark 1.43 and Proposition 1.44, the associated Picard stack is $[G_{A^0}/G_{A^1}] \simeq [G_{\ker \phi}/G_{\operatorname{coker} \phi}]$, which is a Deligne-Mumford stack if and only if $\operatorname{coker} \phi$ is finite.

Now we can define, in this picture, what is a *Deligne-Mumford torus*.

DEFINITION 1.46. A *Deligne-Mumford torus* is a Picard stack over $\operatorname{Spec}(\mathbb{C})$ obtained as a quotient $[G_{A^0}/G_{A^1}]$ for a morphism $\phi: A^0 \rightarrow A^1$ of finitely generated abelian groups, with free kernel and finite cokernel. ■

Note that by this definition, for any finite abelian group G , the stack $\mathcal{B}G$ is a Deligne-Mumford torus. Moreover, every ordinary torus $T = (\mathbb{C}^*)^n$ is a Deligne-Mumford torus. The following characterization shows that this two types of Deligne-Mumford tori are enough to construct every Deligne-Mumford torus.

PROPOSITION 1.47. [39, Proposition 2.6] *For any Deligne-Mumford torus \mathcal{T} there exist a torus T and a finite abelian group G such that \mathcal{T} is isomorphic as a Picard stack to the product $T \times \mathcal{B}G$.*

The idea of the proof is that, if \mathcal{T} is the quotient stack $[G_{A^0}/G_{A^1}]$, then there is an exact sequence of Picard stacks

$$1 \rightarrow \mathcal{B}G \rightarrow \mathcal{T} \rightarrow T \rightarrow 1$$

where $T := G_{A^0}/G_{A^1}$, and this induces a (non-canonical) isomorphism.

1.5. Toric Deligne-Mumford stacks

After introducing the notion of Deligne-Mumford torus, in this section we are ready to give the definition of smooth toric Deligne-Mumford stacks, and to study their first properties. In particular we will study the cases in which the toric Deligne-Mumford stack is canonical, is an orbifold, and finally, via the root constructions, we will give a characterization of general toric Deligne-Mumford stacks as gerbes over an orbifold. The main reference is [39, Section 3].

Throughout this and the next Section, we restrict a bit our conventions: we set $k = \mathbb{C}$. A variety will be a reduced, irreducible scheme. We will always assume that a Deligne-Mumford stack has a coarse moduli *scheme*.

DEFINITION 1.48. A (*smooth*) *toric Deligne-Mumford stack* is a smooth Deligne-Mumford stack \mathcal{X} with an open dense immersion $i: \mathcal{T} \rightarrow \mathcal{X}$ of a Deligne-Mumford torus \mathcal{T} such that the canonical action of \mathcal{T} on itself extends to an action $a: \mathcal{T} \times \mathcal{X} \rightarrow \mathcal{X}$ on the whole \mathcal{X} . A *morphism of toric Deligne-Mumford stacks* $F: \mathcal{X} \rightarrow \mathcal{X}'$ between two toric Deligne-Mumford stacks $\mathcal{X}, \mathcal{X}'$ with Deligne-Mumford tori $\mathcal{T}, \mathcal{T}'$ respectively, is a morphism of stacks which extends a morphism of Deligne-Mumford tori $\mathcal{T} \rightarrow \mathcal{T}'$. ■

Since we will consider only smooth toric Deligne-Mumford stacks, we will omit the word smooth. We consider also the notion of *toric orbifold*, which is a toric Deligne-Mumford stack such that the stabilizers are generically trivial. It can be shown that a toric Deligne-Mumford stack is a toric orbifold if and only if its Deligne-Mumford torus is an ordinary torus, so this definition of toric orbifold coincides with the definitions previously known in literature, as in [60].

- REMARK 1.49. • Note that by the separateness of \mathcal{X} and [39, Proposition 1.2], the action of the Deligne-Mumford torus \mathcal{T} on \mathcal{X} is uniquely determined by the open dense immersion i ;
- A toric variety has a (canonical) structure of toric Deligne-Mumford stack if and only if it is smooth.

△

The following result, which is due just to the properties of the coarse moduli spaces of Deligne-Mumford stacks, shows how toric Deligne-Mumford stacks stand in connection with toric varieties.

PROPOSITION 1.50. [39, Proposition 3.6] *Let \mathcal{X} be a toric Deligne-Mumford stack with Deligne-Mumford torus \mathcal{T} . Let X, T be the coarse moduli spaces of \mathcal{X}, \mathcal{T} , respectively. Then the open dense immersion $i: \mathcal{T} \rightarrow \mathcal{X}$ and the action $a: \mathcal{T} \times \mathcal{X} \rightarrow \mathcal{X}$ induces an open dense immersion $\bar{i}: T \rightarrow X$ and an action $\bar{a}: T \times X \rightarrow X$, which gives X the structure of a simplicial toric variety with torus T .*

Note that all simplicial toric varieties are irreducible, thus [107, Lemma 2.3] and the Proposition above ensure that all toric stacks are irreducible.

Consider the structure morphism $\pi: \mathcal{X} \rightarrow X$ from a toric Deligne-Mumford stack to its coarse moduli space. By [72, Corollary 5.6.1], π induces a bijection on reduced closed substacks. Let D_i for $i = 1, \dots, n$ be the irreducible torus-invariant Weil divisors in X , and denote by $\mathcal{D}_i := \pi^{-1}(D_i)_{red}$ the reduced closed substack with support $\pi^{-1}(D_i)$. Being $D_i \cap X_{sm}$ a Cartier divisor, there exists a positive integer a_i such that $\pi^{-1}(D_i \cap X_{sm}) = a_i(\mathcal{D}_i \cap \pi^{-1}(X_{sm}))$. We will call the a_i 's the *divisor multiplicities* of \mathcal{D}_i in \mathcal{X} .

1.5.1. Canonical toric Deligne-Mumford stacks. Recall that ([39, Definition 4.4]) a *canonical stack* is an irreducible d -dimensional smooth Deligne-Mumford stack \mathcal{X} such that the locus where the structure morphism $\pi: \mathcal{X} \rightarrow X$ to the coarse moduli space is not an isomorphism has dimension $\leq d - 2$. If \mathcal{X} is a canonical stack, the locus where π is an isomorphism is precisely $\pi^{-1}(X_{sm})$, where X_{sm} is the smooth locus of X . Moreover the composition of isomorphisms

$$A^1(X) \xrightarrow{\cong} A^1(X_{sm}) \xrightarrow{\cong} \text{Pic}(X_{sm}) \xrightarrow{\cong} \text{Pic}(\pi^{-1}(X_{sm})) \xrightarrow{\cong} \text{Pic}(\mathcal{X})$$

sends the class of a divisor $[D]$ to the line bundle corresponding to the preimage of the divisor under the structure morphism $\mathcal{O}_{\mathcal{X}}(\pi^{-1}(D)_{red})$.

THEOREM 1.51 (Universal property of canonical smooth Deligne-Mumford stacks). [39, Theorem 4.6] *Consider a canonical smooth Deligne-Mumford stack \mathcal{Y} and denote by $\pi: \mathcal{Y} \rightarrow Y$ its structure morphism to the coarse moduli space. Let $f: \mathcal{X} \rightarrow Y$ be a dominant codimension preserving morphism from an orbifold. Then there exists a unique, up to a unique 2-arrow, morphism $g: \mathcal{X} \rightarrow \mathcal{Y}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathcal{Y} \\ & \searrow f & \downarrow \pi \\ & & Y \end{array}$$

REMARK 1.52. It follows from the universal property that two canonical smooth Deligne-Mumford stacks with isomorphic coarse moduli spaces are actually isomorphic. This can be used to prove that every variety Y with finite quotient singularities is the coarse moduli space of a canonical smooth Deligne-Mumford stack, denoted \mathcal{Y}^{can} , unique up to a rigid isomorphism. In particular, if $Y = Z/G$ as a geometric quotient, with Z a smooth variety and G a group without pseudo-reflections acting with finite stabilizers, then $\mathcal{Y}^{can} = [Y/G]$. This can be applied, for example, to the case of simplicial toric varieties. \triangle

COROLLARY 1.53. [39, Corollary 4.10] *Let $\pi: \mathcal{X} \rightarrow X$ be the structure morphism from a smooth Deligne-Mumford stack to its coarse moduli space. There exists a unique morphism $\mathcal{X} \rightarrow \mathcal{X}^{can}$ through which π factors.*

We want to study the canonical stacks associated with simplicial toric varieties. They are characterized by the following result.

THEOREM 1.54. [39, Theorem 4.11] *Let \mathcal{X}^{can} be the canonical stack associated with a simplicial toric variety X with torus T . Then the action $\bar{a}: T \times X \rightarrow X$ lifts to an action $a^{can}: T \times \mathcal{X}^{can} \rightarrow \mathcal{X}^{can}$ which gives to \mathcal{X}^{can} a natural structure of toric orbifold.*

The construction in the proof of this result is the following: the toric variety X can be written as a geometric quotient $X = Z_\Sigma/G_A$ (see Section 1.6 below), where Z_Σ is an affine space minus a codimension two closed subvariety, and G_A is a torus. Then by Remark 1.52, the canonical stack \mathcal{X}^{can} of \mathcal{X} is isomorphic to the quotient stack $[Z_\Sigma/G_A]$. Note also that its Deligne-Mumford torus is $\mathcal{T}^{can} \simeq [(\mathbb{C}^*)^n/G_A]$, and the restriction of the structure morphism $\pi: \mathcal{X}^{can} \rightarrow X$ to \mathcal{T}^{can} is an isomorphism with T .

The following corollary shows that a similar construction holds for canonical toric Deligne-Mumford stacks: we can realize them as a quotient stack. This shows in particular that the global quotient stack description of a canonical toric Deligne-Mumford stack is related to the geometric quotient description of its coarse moduli space.

COROLLARY 1.55. [39, Corollary 4.13] *Let \mathcal{X} be a canonical toric Deligne-Mumford stack with torus $\mathcal{T} = T$. Let a simplicial toric variety X be its coarse moduli space, and denote by $\Sigma \subset N_\mathbb{Q}$ the fan of X . Then the following hold:*

- (1) *If the rays of Σ generate $N_\mathbb{Q}$, then $\mathcal{X} = [Z_\Sigma/G_A]$ where $G_A = \text{Hom}(A^1(X), \mathbb{C}^*) = \text{Hom}(\text{Pic}(\mathcal{X}), \mathbb{C}^*)$.*
- (2) *The boundary divisor $\mathcal{X} \setminus T$ is simple normal crossing. Denote by \mathcal{D}_i its irreducible components. If the rays of Σ generate $N_\mathbb{Q}$, then each divisor \mathcal{D}_i is isomorphic to $[Z_\Sigma \cap \{x_i = 0\}/G_A]$.*
- (3) *The morphism $\text{Div}^T(X) \rightarrow A^1(X) \xrightarrow{\pi^*} \text{Pic}(\mathcal{X})$ sends e_i to $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$.*

REMARK 1.56. With the same assumptions, we can note the following:

- If the rays of Σ generates $N_\mathbb{Q}$, there is a short exact sequence

$$0 \rightarrow M \rightarrow \text{Div}^T(X) \rightarrow \text{Pic}(\mathcal{X}) \rightarrow 0,$$

where M is the character group of T .

- Each divisor \mathcal{D}_i is Cartier, and corresponds to a line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$, with a canonical section s_i . The line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$ is associated with the representation $G_A \rightarrow$

$G_{\text{Div}^T(X)} = (\mathbb{C}^*)^n \xrightarrow{p_i} \mathbb{C}^*$, where p_i is the i -th projection, and the canonical section s_i is the i -th coordinate in Z_Σ .

- If \mathcal{X} is a canonical toric Deligne-Mumford stack, all divisor multiplicities are 1.

△

1.5.2. Toric orbifolds. Consider now a toric Deligne-Mumford stack \mathcal{X} with generically trivial stabilizer, i.e., a toric orbifold, with torus T . Let X be its coarse moduli space and $\pi: \mathcal{X} \rightarrow X$ the structure morphism. By Proposition 1.50 and Theorem 1.54, the associated canonical stack \mathcal{X}^{can} has a structure of toric orbifold, with coarse moduli space X . Let $\pi^{\text{can}}: \mathcal{X}^{\text{can}} \rightarrow X$ be the morphism to the coarse moduli space. By the universal property of canonical stacks (Theorem 1.51) there exists a unique $f: \mathcal{X} \rightarrow \mathcal{X}^{\text{can}}$ which factorizes π through π^{can} . By [39, Proposition 5.1], f is a morphism of toric Deligne-Mumford stacks. Notice that it induces an isomorphism $f|_T: T \xrightarrow{\sim} \mathcal{T}^{\text{can}}$.

Denote by $\tilde{\mathcal{D}}_1, \dots, \tilde{\mathcal{D}}_n$ the irreducible components of the boundary divisor $\mathcal{X}^{\text{can}} \setminus T$ (see Corollary 1.55), and $\tilde{\mathcal{D}} = (\tilde{\mathcal{D}}_1, \dots, \tilde{\mathcal{D}}_n)$.

THEOREM 1.57. [39, Theorem 5.2])

- (1) Consider a simplicial toric variety X with torus T , and let Σ be its fan. Choose a positive integer a_i for every ray $\rho_i \in \Sigma(1)$, and denote $\vec{a} = (a_1, \dots, a_n)$. Then the root stack $\sqrt[n]{\tilde{\mathcal{D}}/\mathcal{X}^{\text{can}}}$ has a unique structure of toric orbifold with torus T such that the canonical morphism $r: \sqrt[n]{\tilde{\mathcal{D}}/\mathcal{X}^{\text{can}}} \rightarrow \mathcal{X}^{\text{can}}$ is a morphism of toric Deligne-Mumford stacks with divisor multiplicities \vec{a} .
- (2) If \mathcal{X} is a toric orbifold with coarse moduli space X and divisor multiplicities \vec{a} , then \mathcal{X} is naturally isomorphic, as a toric Deligne-Mumford stack, to $\sqrt[n]{\tilde{\mathcal{D}}/\mathcal{X}^{\text{can}}}$.

As a consequence, in the assumption of (2), the reduced closed substack $\mathcal{X} \setminus T$ is a simple normal crossing divisor. Moreover there exists a morphism of short exact sequences

$$(9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\vec{a}} & \mathbb{Z}^n & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}_{a_i} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Pic}(\mathcal{X}^{\text{can}}) & \xrightarrow{f^*} & \text{Pic}(\mathcal{X}) & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}_{a_i} \longrightarrow 0 \end{array}$$

where the first vertical morphism sends $e_i \mapsto \mathcal{O}_{\mathcal{X}^{\text{can}}}(\tilde{\mathcal{D}}_i)$, and the second $e_i \mapsto \mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$.

1.5.3. Characterization of toric Deligne-Mumford stacks. In this section we give a structure theorem for toric Deligne-Mumford stacks, characterizing them as gerbes over their rigidifications. First we recall what a rigidification of a Deligne-Mumford stack is. Intuitively, the rigidification of \mathcal{X} by its generic stabilizer G^3 is a stack where the objects are the same and the automorphism group of an object x is the quotient $\text{Aut}_{\mathcal{X}}(x)/G$. Rigidifications can be defined for any central subgroup of the generic stabilizer, but we are not interested in this. For the general construction we refer to [3, Appendix A] (see also [1, Section 5.1]).

³The generic stabilizer is actually defined as the union, inside the inertia stack $\mathcal{I}(\mathcal{X})$, of all the components of maximal dimension, and is a subsheaf of groups of $\mathcal{I}(\mathcal{X})$.

We will call $r: \mathcal{X} \rightarrow \mathcal{X}^{rig}$ the rigidification of \mathcal{X} . We recall the main properties: \mathcal{X}^{rig} is an orbifold with the same coarse moduli space of \mathcal{X} , if \mathcal{X} is an orbifold then \mathcal{X}^{rig} is \mathcal{X} , and the morphism r makes \mathcal{X} into a gerbe over \mathcal{X}^{rig} .

Let now \mathcal{X} be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T} \simeq T \times \mathcal{B}G$ and coarse moduli space X . Let \mathcal{X}^{rig} be its rigidification, which is an orbifold with coarse moduli space X . By [39, Lemma 3.8], the generic stabilizer of \mathcal{X} is isomorphic to $G \times \mathcal{X}$, and by [39, Lemma 6.23] \mathcal{X}^{rig} has a unique structure of toric orbifold with torus T such that the morphism $\mathcal{T} \rightarrow T$ of Deligne-Mumford torus induces a morphism $r: \mathcal{X} \rightarrow \mathcal{X}^{rig}$ of toric Deligne-Mumford stacks. Moreover, there is a morphism of toric Deligne-Mumford stacks $f^{rig}: \mathcal{X}^{rig} \rightarrow \mathcal{X}^{can}$ which factorizes the natural morphism $f: \mathcal{X} \rightarrow \mathcal{X}^{can}$ through r . Note also that r is étale, thus the divisor multiplicities of \mathcal{X} and \mathcal{X}^{rig} are the same.

THEOREM 1.58. [39, Theorem 6.25] *If \mathcal{Y} is a toric orbifold with Deligne-Mumford torus T , and $\mathcal{X} \rightarrow \mathcal{Y}$ is an essentially trivial G -gerbe, then there exists on \mathcal{X} a unique structure of toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $T \times \mathcal{B}G$, such that the gerbe morphism $\mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of toric Deligne-Mumford stacks. Conversely, any toric Deligne-Mumford stack \mathcal{X} with Deligne-Mumford torus $\mathcal{T} \simeq T \times \mathcal{B}G$ is an essentially trivial G -gerbe $r: \mathcal{X} \rightarrow \mathcal{X}^{rig}$ over its rigidification.*

COROLLARY 1.59. [39, Corollary 6.27] *Consider a toric Deligne-Mumford stack \mathcal{X} with Deligne-Mumford torus $\mathcal{T} \simeq T \times \mathcal{B}G$, and let G be a product $\prod_{j=1}^l \mu_{b_j}$. Then there exist l line bundles $\mathcal{L}_j \in \text{Pic}(\mathcal{X}^{rig})$ such that \mathcal{X} is isomorphic as a G -banded gerbe over \mathcal{X}^{rig} to*

$$\sqrt[b_1]{\mathcal{L}_1 / \mathcal{X}^{rig}} \times_{\mathcal{X}^{rig}} \dots \times_{\mathcal{X}^{rig}} \sqrt[b_l]{\mathcal{L}_l / \mathcal{X}^{rig}},$$

and the classes $[\mathcal{L}_j] \in \text{Pic}(\mathcal{X}^{rig})/b_j \text{Pic}(\mathcal{X}^{rig})$ are unique. Moreover, the closed substack $\mathcal{X} \setminus \mathcal{T}$ is a simple normal crossing divisor.

The above corollary imply also that there is a morphism of short exact sequences

$$(10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^l & \xrightarrow{\cdot \bar{b}} & \mathbb{Z}^l & \longrightarrow & \bigoplus_{j=1}^l \mathbb{Z}_{b_j} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Pic}(\mathcal{X}^{rig}) & \xrightarrow{r^*} & \text{Pic}(\mathcal{X}) & \longrightarrow & \bigoplus_{j=1}^l \mathbb{Z}_{b_j} \longrightarrow 0 \end{array}$$

where the first vertical morphism sends $e_j \mapsto \mathcal{L}_j$, and the second sends $e_j \mapsto \mathcal{L}_j^{1/b_j}$.

1.6. Toric Deligne-Mumford stacks and stacky fans

Here we describe an analog of the construction in Section 1.3 for toric Deligne-Mumford stacks, due to Fantechi, Mann and Nironi ([39]).

Consider a toric Deligne-Mumford stack \mathcal{X} with coarse moduli space X . By Proposition 1.50, X is a simplicial toric variety. Let Σ be its fan, and assume the rays generate $N_{\mathbb{Q}}$. From what we saw in the previous section, X is the geometric quotient $Z_{\Sigma}/G_{A^1(X)}$, where $G_{A^1(X)} = \text{Hom}(A^1(X), \mathbb{C}^*)$.

Set $G_{\mathcal{X}} := \text{Hom}(\text{Pic}(\mathcal{X}), \mathbb{C}^*)$. Consider the composition of morphisms $\mathbb{Z}^n \rightarrow \text{Pic}(\mathcal{X}^{rig}) \rightarrow \text{Pic}(\mathcal{X})$. Applying $\text{Hom}(\cdot, \mathbb{C}^*)$, we obtain a morphism of diagonalizable groups $G_{\mathcal{X}} \rightarrow G_{A^1(X)}$,

and via this $G_{\mathcal{X}}$ acts on Z_{Σ} . We can consider the quotient stack $[Z_{\Sigma}/G_{\mathcal{X}}]$. Note that the quotient stack $[(\mathbb{C}^*)^n/G_{\mathcal{X}}]$ is open and dense in $[Z_{\Sigma}/G_{\mathcal{X}}]$, and is a Deligne-Mumford torus. Moreover, the natural action of $(\mathbb{C}^*)^n$ on Z_{Σ} extends the action of $(\mathbb{C}^*)^n$ on itself, thus there is a stack morphism

$$a: [(\mathbb{C}^*)^n/G_{\mathcal{X}}] \times [Z_{\Sigma}/G_{\mathcal{X}}] \rightarrow [Z_{\Sigma}/G_{\mathcal{X}}]$$

that extends the action of $[(\mathbb{C}^*)^n/G_{\mathcal{X}}]$ on itself. Thus by [39, Proposition 3.3], a is an action and then $[Z_{\Sigma}/G_{\mathcal{X}}]$ is a toric Deligne-Mumford stack.

The main result of the section is the following theorem which characterizes toric Deligne-Mumford stacks as the quotients stacks constructed as described above.

THEOREM 1.60. [39, Theorem 7.7] *Let \mathcal{X} be a toric Deligne-Mumford stack with coarse moduli space X . Let Σ be the fan of X , and assume its rays generate $N_{\mathbb{Q}}$. Then \mathcal{X} is naturally isomorphic, as a toric stack, to $[Z_{\Sigma}/G_{\mathcal{X}}]$, where $G_{\mathcal{X}} := \text{Hom}(\text{Pic}(\mathcal{X}), \mathbb{C}^*)$.*

Note that this Theorem, when \mathcal{X} is a canonical toric Deligne-Mumford stack \mathcal{X}^{can} , reduces to Corollary 1.55. For the general case, the proof follows from the following two facts.

LEMMA 1.61. [39, Lemma 7.1] *Consider a scheme Z and an abelian group scheme G over \mathbb{C} that acts on Z , such that the quotient stack $[Z/G]$ is a Deligne-Mumford stack. Let $(\vec{\mathcal{L}}, \vec{s}) = ((\mathcal{L}_1, s_1), \dots, (\mathcal{L}_n, s_n))$ be n pairs, each one given by a line bundle and a global sections on $[Z/G]$, and let $\vec{\chi} = (\chi_1, \dots, \chi_n)$ be the representations associated with them⁴. Let also $\vec{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{>0}^n$ be n positive integers.*

- (1) *The \vec{k} -root $\sqrt[\vec{k}]{(\vec{\mathcal{L}}, \vec{s})/[Z/G]}$ is isomorphic to $[\tilde{Z}/\tilde{G}]$ where \tilde{Z} and \tilde{G} are defined by the cartesian diagrams*

$$\begin{array}{ccc} \tilde{Z} & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow (\cdot)^{\vec{a}} \\ Z & \xrightarrow{\vec{s}} & \mathbb{A}^n, \end{array} \quad \begin{array}{ccc} \tilde{G} & \longrightarrow & \mathbb{G}_m^n \\ \phi \downarrow & & \downarrow (\cdot)^{\vec{k}} \\ G & \xrightarrow{\vec{\chi}} & \mathbb{G}_m^n. \end{array}$$

- (2) *The \vec{a} -root $\sqrt[\vec{a}]{\vec{\mathcal{L}}/[Z/G]}$ is isomorphic to $[Z/\tilde{G}]$ where \tilde{G} is defined above. The action of \tilde{G} on Z is given via ϕ .*

REMARK 1.62. Note that by construction of the action of \tilde{G} on Z , the kernel of ϕ acts trivially on Z . Moreover, $\ker \phi$ is of the form $\prod_{i=1}^n \mu_{k_i}$, thus $[Z/\tilde{G}]$ is a $\prod_{i=1}^n \mu_{k_i}$ -banded gerbe over $[Z/G]$. \triangle

1.6.1. Stacky fans and associated Deligne-Mumford stacks. In this section we present a combinatorial approach, due to Borisov, Chen and Smith [18], to the theory of toric stacks; as one can associate with a toric variety the combinatorial datum represented by a fan, for a toric stack one can introduce a new kind of combinatorial datum, called a *stacky fan*.

⁴Recall that a line bundle on $[Z/G]$ is uniquely determined by the choice of a character χ of G .

1.6.1.1. *Gale duality with torsion.* Here we follow the presentation in [18, Section 2] of the *generalized Gale duality*. We start by recalling the classical Gale duality (see [110, Theorem 6.14]). Given n vectors b_1, \dots, b_n which span \mathbb{Q}^d , there exists a dual configuration $[a_1 \dots a_n] \in \mathbb{Q}^{(n-d) \times n}$ that gives a short exact sequence

$$(11) \quad 0 \rightarrow \mathbb{Q}^d \xrightarrow{[b_1 \dots b_n]^T} \mathbb{Q}^n \xrightarrow{[a_1 \dots a_n]} \mathbb{Q}^{n-d} \rightarrow 0.$$

The vectors $\{a_1, \dots, a_n\}$ are uniquely determine up to a linear transformation in \mathbb{Q}^{n-d} .

This duality is important in the study of smooth toric varieties ([45, Section 3.4]): take a fan Σ with n rays such that the corresponding toric variety X_Σ is smooth. If $N \simeq \mathbb{Z}^d$, the minimal lattice points b_1, \dots, b_n generating the rays give a map $\beta: \mathbb{Z}^n \rightarrow N$. Tensoring with \mathbb{Q} and applying Gale duality, we get a dual configuration $\{a_1, \dots, a_n\}$. Because X_Σ is smooth, $a_i \in \mathbb{Z}^{n-d}$, and the a_i are determine up to unimodular transformations. They determine a map $\beta^\vee: (\mathbb{Z}^n)^* \rightarrow \mathbb{Z}^{n-d} \simeq \text{Pic}(X_\Sigma)$, and the short exact sequence (11) becomes the sequence (7), which characterizes the Picard group of X_Σ . Here we denoted $(\cdot)^* := \text{Hom}(\cdot, \mathbb{Z})$.

In [18, Section 2] the authors extend this construction to a larger class of maps. In particular, let N be a finitely generated abelian group and $\beta: \mathbb{Z}^n \rightarrow N$ be a group homomorphism. Define the dual map $\beta^\vee: (\mathbb{Z}^n)^* \rightarrow DG(\beta)$ as follows. Take projective resolutions E^\bullet and F^\bullet for \mathbb{Z}^n and N respectively. By [108, Theorem 2.2.6], β lifts to a morphism $E^\bullet \rightarrow F^\bullet$, and by [108, 1.5.8], there is a short exact sequence of cochain complexes $0 \rightarrow F^\bullet \rightarrow \text{Cone}(\beta) \rightarrow E^\bullet[1] \rightarrow 0$, where $\text{Cone}(\beta)$ is the mapping cone. being E^\bullet projective, we have the exact sequence of cochain complexes

$$0 \rightarrow E^\bullet[1]^* \rightarrow \text{Cone}(\beta)^* \rightarrow (F^\bullet)^* \rightarrow 0$$

and this induces a long exact sequence in cohomology that contains the exact sequence

$$(12) \quad N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \rightarrow H^1(\text{Cone}(\beta)^*) \rightarrow \text{Ext}_{\mathbb{Z}}^1(N, \mathbb{Z}) \rightarrow 0.$$

Define $DG(\beta) := H^1(\text{Cone}(\beta)^*)$ and $\beta^\vee: (\mathbb{Z}^n)^* \rightarrow DG(\beta)$ to be the second map in (12). By this definition, it is obvious that the construction is natural.

There is also an explicit description of β^\vee . If d is the rank of N , one can choose a projective resolution of N of the form $0 \rightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \rightarrow 0$, where Q is an integer matrix. Then $\beta: \mathbb{Z}^n \rightarrow N$ lifts to a map $\mathbb{Z}^n \xrightarrow{B} \mathbb{Z}^{d+r}$. Then $\text{Cone}(\beta)$ is the complex $0 \rightarrow \mathbb{Z}^{n+r} \xrightarrow{[BQ]} \mathbb{Z}^{d+r} \rightarrow 0$, hence $DG(\beta) = (\mathbb{Z}^{n+r})^* / \text{Im}([BQ]^*)$, and β^\vee is the composition of the inclusion map $(\mathbb{Z}^n)^* \rightarrow (\mathbb{Z}^{n+r})^*$ with the quotient map $(\mathbb{Z}^{n+r})^* \rightarrow DG(\beta)$.

We give here a property of this generalized Gale dual that will be useful in the following.

LEMMA 1.63. *A morphism of short exact sequences*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^{n_2} & \longrightarrow & \mathbb{Z}^{n_3} \longrightarrow 0 \\ & & \downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow \beta_3 \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 \longrightarrow 0, \end{array}$$

in which the columns have finite cokernel, induces a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathbb{Z}^{n_1})^* & \longrightarrow & (\mathbb{Z}^{n_2})^* & \longrightarrow & (\mathbb{Z}^{n_3})^* \longrightarrow 0 \\ & & \downarrow \beta_1^\vee & & \downarrow \beta_2^\vee & & \downarrow \beta_3^\vee \\ 0 & \longrightarrow & DG(\beta_1) & \longrightarrow & DG(\beta_2) & \longrightarrow & DG(\beta_3) \longrightarrow 0. \end{array}$$

1.6.1.2. Stacky fans.

DEFINITION 1.64. (Stacky fan, [18, Section 3]) A *stacky fan* is a triple $\Sigma := (N, \Sigma, \beta)$ where

- N is a finitely generated (in general not free) abelian group of rank d . Denote by \bar{N} the lattice generated by N in the d -dimensional vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, and by $b \mapsto \bar{b}$ the natural map $N \rightarrow \bar{N}$.
- Σ is a rational simplicial fan in $N_{\mathbb{Q}}$. Denote by ρ_1, \dots, ρ_n the rays in $\Sigma(1)$, and assume that they generate $N_{\mathbb{Q}}$.
- $\beta: \mathbb{Z}^n \rightarrow N$ is a homomorphism with finite cokernel, determined by n elements $b_i \in N$ such that \bar{b}_i generates the cone ρ_i for $i = 1, \dots, n$.

■

Now we give the construction of a Deligne-Mumford stack associated with a stacky fan ([18, Section 3]). The construction is very similar to the presentation of a toric variety as a geometric quotient, but uses the generalized Gale duality we presented in the previous section.

Consider the quasi-affine variety $Z_{\Sigma} \subset \mathbb{A}^n$ defined for toric varieties (Section 1.3). Then Z_{Σ} has an action of $G_{\Sigma} = \text{Hom}(DG(\beta), \mathbb{C}^*)$ constructed as follows. Take the Gale dual $\beta^\vee: (\mathbb{Z}^n)^* \rightarrow N$, and apply $\text{Hom}(\cdot, \mathbb{C}^*)$. This gives a morphism $G_{\Sigma} \rightarrow (\mathbb{C}^*)^n$. Composing with the natural action of $(\mathbb{C}^*)^n$ on \mathbb{A}^n , we obtain an action of G_{Σ} on \mathbb{A}^n , and one can show that Z_{Σ} is invariant, thus G_{Σ} acts on Z_{Σ} .

Define $\mathcal{X}_{\Sigma} := [Z_{\Sigma}/G_{\Sigma}]$. By [72, Remark 10.13.2], Since Z_{Σ} is smooth and separated, \mathcal{X}_{Σ} is a smooth separated algebraic stack. Since the action of G_{Σ} on Z_{Σ} is such that the stabilizers are finite by [18, Lemma 3.1], \mathcal{X}_{Σ} is a smooth Deligne-Mumford stack. [18, Proposition 3.7] says moreover that \mathcal{X}_{Σ} has X_{Σ} as a coarse moduli space.

REMARK 1.65. Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan and for any $i = 1, \dots, n$ let v_i be the unique generator of $\rho_i \cap (N/N_{\text{tor}})$, where N_{tor} is the torsion part of N . Denote by β^{rig} the composition $\mathbb{Z}^n \xrightarrow{\beta} N \rightarrow N/N_{\text{tor}}$. For each i there exists a unique $a_i \in \mathbb{Z}_{>0}$ such that $\beta^{\text{rig}}(e_i) = a_i v_i$. Let $\Sigma^{\text{rig}} := (N/N_{\text{tor}}, \Sigma, \beta^{\text{rig}})$. There is a unique morphism $\beta^{\text{can}}: \mathbb{Z}^n \rightarrow N/N_{\text{tor}}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\beta} & N \\ \text{diag}(a_1, \dots, a_n) \downarrow & \searrow \beta^{\text{rig}} & \downarrow \\ \mathbb{Z}^n & \xrightarrow{\beta^{\text{can}}} & N/N_{\text{tor}}. \end{array}$$

Define also $\Sigma^{\text{can}} := (N/N_{\text{tor}}, \Sigma, \beta^{\text{can}})$.

△

LEMMA 1.66. [39, Lemma 7.15] *The stack \mathcal{X}_{Σ} :*

- (1) *is a toric Deligne-Mumford stack.*
- (2) *is a toric orbifold if and only if N is free.*
- (3) *is canonical if and only if $\Sigma = \Sigma^{\text{can}}$.*

In particular we have $\mathcal{X}_{\Sigma}^{\text{rig}} \simeq \mathcal{X}_{\Sigma^{\text{rig}}}$, and $\mathcal{X}_{\Sigma}^{\text{can}} \simeq \mathcal{X}_{\Sigma^{\text{can}}}$.

Moreover, one can deduce that if $\Sigma = (N, \Sigma, \beta)$ is a stacky fan, then $\text{Pic}(\mathcal{X}_{\Sigma}) \simeq DG(\beta)$, the Gale dual of the map β , and thus $G_{\Sigma} \simeq G_{\mathcal{X}}$.

Up to now we have shown that every stacky fan gives rise to a toric Deligne-Mumford stack. The following result by Fantechi, Mann and Nironi shows that also the converse is true.

THEOREM 1.67. *Consider a toric Deligne-Mumford stack \mathcal{X} with coarse moduli space X . Let Σ be the fan of X in $N_{\mathbb{Q}}$, and assume that the rays of Σ span $N_{\mathbb{Q}}$. Then there is a finitely generated abelian group N of rank $\dim_{\mathbb{Q}} N_{\mathbb{Q}}$ and a map $\beta: \mathbb{Z}^n \rightarrow N$ such that $\mathcal{X}_{(N, \Sigma, \beta)} \simeq \mathcal{X}$ as toric Deligne-Mumford stacks.*

1.6.2. Closed and open substacks. In this section we show how the combinatorial data of the stacky fan Σ encode certain substacks of \mathcal{X}_{Σ} . We use this result to give a description of the inertia stack. We follow [18, Section 3].

Consider a stacky fan $\Sigma = (N, \Sigma, \beta)$, and fix a cone $\sigma \in \Sigma$. Define N_{σ} to be the subgroup of N generated by $\{b_i | \rho_i \subset \sigma\}$. Define $N(\sigma) := N/N_{\sigma}$, then the quotient map induces a surjection $N_{\mathbb{Q}} \rightarrow N(\sigma)_{\mathbb{Q}}$. Recall that the *quotient fan* is $\Sigma/\sigma \subset N(\sigma)_{\mathbb{Q}}$, defined by

$$\Sigma/\sigma := \{\tilde{\tau} = \tau + (N_{\sigma})_{\mathbb{Q}} | \sigma \subset \tau \text{ and } \tau \in \Sigma\}.$$

Recall that $\text{link}(\sigma) = \{\tau | \tau + \sigma \in \Sigma, \tau \cap \sigma = 0\}$. Given a ray $\rho_i \in \text{link}(\sigma)$, we write $\tilde{\rho}_i$ for that ray in Σ/σ , and call \tilde{b}_i for the image of b_i in the quotient $N(\sigma)$.

We want the previous construction to give a stacky fan, so, as it is not true in general, we need to assume that the rays $\tilde{\rho}_i$ generate $N(\sigma)_{\mathbb{Q}}$. Observe that it suffices to assume that Σ is a complete fan to ensure this condition for every $\sigma \in \Sigma$. It remains to define an analog of the map β . Let $l := |\text{link}(\sigma)|$ be the cardinality of the link, and define

$$\beta(\sigma): \mathbb{Z}^l \rightarrow N(\sigma)$$

to be the map determined by the elements $\{\tilde{b}_i | \rho_i \in \text{link}(\sigma)\}$. Then we define a *quotient stacky fan* to be the stacky fan $\Sigma/\sigma = (N(\sigma), \Sigma/\sigma, \beta(\sigma))$. With this stacky fan we can associate the toric Deligne-Mumford stack $\mathcal{X}_{\Sigma/\sigma}$.

PROPOSITION 1.68. [18, Proposition 4.2] *$\mathcal{X}_{\Sigma/\sigma}$ is a closed substack of \mathcal{X}_{Σ} .*

The proof aim to show that certain quotient stack is actually the stack $\mathcal{X}_{\Sigma/\sigma}$. We don't give here the proof, but we recall the construction of the quotient stack.

Define $W(\sigma)$ to be the closed subvariety in Z_{Σ} defined by $\{z_i = 0\}$. Note that $W(\sigma)$ is G_{Σ} -invariant, as it is a coordinate subspace. Thus we can consider the quotient stack, and the Proposition states

$$\mathcal{X}_{\Sigma/\sigma} \simeq [W(\sigma)/G_{\sigma}].$$

Note that, as pointed out in [61, Remark 5.2], the original proof given in [18] has a gap. A complete proof of the result can be found in [61, Section 5.1].

One can show that, in particular for the choice $\sigma = \rho_i$, one obtains a realization of the torus-invariant divisor \mathcal{D}_i :

$$\mathcal{D}_i \simeq \mathcal{X}_{\Sigma/\rho_i}.$$

We can use these results to give a characterization of the inertia stack $\mathcal{I}(\mathcal{X}_{\Sigma})$. Recall that if \mathcal{X} is a quotient stack of the form $[Z/G]$, then $\mathcal{I}(\mathcal{X}) = \bigsqcup_{g \in G} [Z^g/G]$, where Z^g is the fixed locus of Z with respect to the element $g \in G$ (see for example [18, Section 4]).

After fixing a stacky fan $\Sigma = (N, \Sigma, \beta)$, for every maximal cone $\sigma \in \Sigma$ define the set

$$\text{Box}(\sigma) = \{v \in N \mid \bar{v} = \sum_{\rho_i \subset \sigma} q_i \bar{b}_i \text{ for } 0 \leq q_i < 1\}.$$

Note that $\text{Box}(\sigma)$ is in one-to-one correspondence with the elements in the finite group $N(\sigma)$. Define $\text{Box}(\Sigma) = \bigcup_{\sigma \in \Sigma_{\max}} \text{Box}(\sigma)$, and for every $v \in N$ call $\sigma(v)$ the unique minimal cone containing \bar{v} .

THEOREM 1.69. [61, Lemma 4.6, Theorem 4.7] *If Σ is a complete fan, the elements of $\text{Box}(\Sigma)$ are in one-to-one correspondence with the elements $g \in G_{\Sigma}$ which fix a point in Z_{Σ} , and we have*

$$\mathcal{X}_{\Sigma/\sigma(v)} \simeq [Z_{\Sigma}^g/G_{\Sigma}].$$

Moreover, we can characterize the inertia stack as

$$\mathcal{I}(\mathcal{X}_{\Sigma}) = \bigsqcup_{v \in \text{Box}(\Sigma)} \mathcal{X}_{\Sigma/\sigma(v)}.$$

Viewing a d -dimensional cone $\sigma \in \Sigma$ as the fan consisting of the cone σ and all its faces, we can identify σ with an open substack of $\mathcal{X}(\Sigma)$. Let $\beta_{\sigma}: \mathbb{Z}^d \rightarrow N$ the group homomorphism determined by the set $\{b_i \mid \rho_i \subseteq \sigma\}$. The induced stacky fan σ is the triple $(N, \sigma, \beta_{\sigma})$.

PROPOSITION 1.70. [18, Proposition 4.3] *Let σ be a d -dimensional cone in the fan Σ . Then $\mathcal{X}(\sigma)$ is an open substack of $\mathcal{X}(\Sigma)$, whose coarse moduli scheme is $U_{\sigma} = \text{Spec}(\mathbb{C}[\sigma^{\vee} \cap M])$.*

REMARK 1.71. By varying the d -dimensional cones σ of Σ , the open substacks $\mathcal{X}(\sigma)$ form an open cover of $\mathcal{X}(\Sigma)$. \triangle

CHAPTER 2

Infinite dimensional Lie algebras and representation theory

In this Chapter we present some material about infinite dimensional Lie algebras and their representations that we shall need in the chapters to come. We are mainly interested in introducing the infinite dimensional Heisenberg algebra and its generalizations called *lattice Heisenberg algebras*, in particular the Heisenberg algebra \mathcal{H}_Ω associated with a Dynkin diagram of type A_{k-1} , and in studying their “simplest” representation, namely the Fock space. This is done in Section 2.1. Then we need to define and give some properties of the *affine Kac-Moody algebras* $\widehat{\mathfrak{sl}}_k$ associated with an extended Dynkin diagram of type \hat{A}_{k-1} . In Section 2.2 we introduce the special linear algebra \mathfrak{sl}_k and the affine $\widehat{\mathfrak{sl}}_k$, study how they are related, and give some elements of the representation theory of $\widehat{\mathfrak{sl}}_k$. In the last Section we discuss how representations of \mathcal{H}_Ω induces representations of $\widehat{\mathfrak{sl}}_k$ via the so-called Frenkel-Kac construction.

2.1. Heisenberg algebras

This Section collects some elements about the theory of infinite-dimensional Heisenberg algebras. In particular we introduce the notion of lattice Heisenberg algebras and show how the latter generalize the usual infinite dimensional Heisenberg algebra and the Heisenberg algebra \mathcal{H}_Ω associated with a Dynkin diagram of type A_{k-1} . Then we give the notion of Fock space for a general lattice Heisenberg algebra, and see what it is in the simplest cases. There is a lot of literature about this theory, for example [63]; here we follow [76, Section 1]. Finally, following [30], we give the notion of Whittaker vector for representations of lattice Heisenberg algebras.

2.1.1. Definition of lattice Heisenberg algebras. Let $\mathbb{C} \subseteq \mathbb{F}$ be an extension field of \mathbb{C} . Let \mathbf{L} be a *lattice*, that is, a finite rank free abelian group equipped with a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{L}} : \mathbf{L} \times \mathbf{L} \rightarrow \mathbb{Z}$. Fix a basis $\gamma_1, \dots, \gamma_d$ of \mathbf{L} .

DEFINITION 2.1. The *lattice Heisenberg algebra* $\mathcal{H}_{\mathbb{F}, \mathbf{L}}$ associated with \mathbf{L} is the infinite-dimensional Lie algebra over \mathbb{F} generated by \mathfrak{q}_m^i , for $m \in \mathbb{Z} \setminus \{0\}$ and $i \in \{1, \dots, d\}$, and the *central element* \mathfrak{c} satisfying the relations

$$(13) \quad \begin{cases} [\mathfrak{q}_m^i, \mathfrak{c}] = 0 & \text{for any } m \in \mathbb{Z} \setminus \{0\}, i \in \{1, \dots, d\} , \\ [\mathfrak{q}_m^i, \mathfrak{q}_n^j] = m\delta_{m,-n}\langle \gamma_i, \gamma_j \rangle_{\mathbf{L}} \mathfrak{c} & \text{for any } m, n \in \mathbb{Z} \setminus \{0\}, i, j \in \{1, \dots, d\} . \end{cases}$$

■

For any element $v \in \mathbf{L}$, we may define the element $\mathfrak{q}_m^v \in \mathcal{H}_{\mathbb{F}, \mathbf{L}}$ by linearity. Set

$$(14) \quad \mathcal{H}_{\mathbb{F}, \mathbf{L}}^+ := \bigoplus_{\substack{m > 0 \\ i \in \{1, \dots, d\}}} \mathbb{F} \mathfrak{q}_m^i \quad \text{and} \quad \mathcal{H}_{\mathbb{F}, \mathbf{L}}^- := \bigoplus_{\substack{m < 0 \\ i \in \{1, \dots, d\}}} \mathbb{F} \mathfrak{q}_m^i.$$

Let us denote by $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}})$ (resp. $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}}^{\pm})$) the *universal enveloping algebra* of $\mathcal{H}_{\mathbb{F}, \mathbf{L}}$ (resp. $\mathcal{H}_{\mathbb{F}, \mathbf{L}}^{\pm}$), i.e., the unital associative algebra generated by $\mathcal{H}_{\mathbb{F}, \mathbf{L}}$ (resp. $\mathcal{H}_{\mathbb{F}, \mathbf{L}}^{\pm}$).

EXAMPLE 2.2. The *Heisenberg algebra* $\mathcal{H}_{\mathbb{F}}$ is simply the lattice Heisenberg algebra associated with the lattice $\mathbf{L} := \mathbb{Z}$ with $\langle \cdot, \cdot \rangle_{\mathbf{L}}$ defined by the multiplication between integers. In this case, $\mathcal{H}_{\mathbb{F}}$ is the infinite-dimensional Lie algebra over \mathbb{F} generated by \mathfrak{p}_m , $m \in \mathbb{Z} \setminus \{0\}$ and the central element \mathfrak{c} satisfying the relations

$$\begin{cases} [\mathfrak{p}_m, \mathfrak{c}] = 0 & \text{for any } m \in \mathbb{Z} \setminus \{0\} , \\ [\mathfrak{p}_m, \mathfrak{p}_n] = m\delta_{m, -n} \mathfrak{c} & \text{for any } m, n \in \mathbb{Z} \setminus \{0\} . \end{cases}$$

We define $\mathcal{H}_{\mathbb{F}}^{\pm}$ and the corresponding universal enveloping algebras as before. \triangle

EXAMPLE 2.3. Consider the lattice $\mathbf{L} := \mathbb{Z}^k$, with the nondegenerate symmetric bilinear form $\langle v, w \rangle_{\mathbf{L}} = \sum_{i=1}^k \langle v_i, w_i \rangle_{\mathbb{Z}}$, where the bilinear form on \mathbb{Z} is the multiplication of integers, as in the previous example. We call the lattice Heisenberg algebra over \mathbb{F} associated to \mathbf{L} the *rank k Heisenberg algebra* over \mathbb{F} , and we denote it $\mathcal{H}_{\mathbb{F}}^k$. It is generated by elements \mathfrak{p}_m^i , $m \in \mathbb{Z} \setminus \{0\}$, $i = 1, \dots, k$, and the central element \mathfrak{c} satisfying the relations

$$\begin{cases} [\mathfrak{p}_m^i, \mathfrak{c}] = 0 & \text{for any } m \in \mathbb{Z} \setminus \{0\}, i = 1, \dots, k , \\ [\mathfrak{p}_m^i, \mathfrak{p}_n^j] = m\delta_{i,j}\delta_{m, -n} \mathfrak{c} & \text{for any } m, n \in \mathbb{Z} \setminus \{0\}, i, j = 1, \dots, k . \end{cases}$$

This lattice Heisenberg algebra can be realized as the sum of k commuting copies of the Heisenberg algebra of the previous example, identifying all the central elements in each copy. Again we define the universal enveloping algebra and $(\mathcal{H}_{\mathbb{F}}^k)^{\pm}$ as before. \triangle

EXAMPLE 2.4. Let $k \geq 2$ be an integer and \mathfrak{Q} the *root lattice* of type A_{k-1} . Let $\mathcal{H}_{\mathbb{F}, \mathfrak{Q}}$ be the lattice Heisenberg algebra over \mathbb{F} associated to \mathfrak{Q} . We shall call it the *Heisenberg algebra of type A_{k-1}* over \mathbb{F} . Recall that the root lattice \mathfrak{Q} of type A_{k-1} can be realized as a sublattice of \mathbb{Z}^k : if e_1, \dots, e_k is the standard basis, then the elements $\gamma_i := e_i - e_{i+1}$ for $i = 1, \dots, k-1$ form a standard basis of \mathfrak{Q} and correspond to the simple roots (see Remark 2.10 below). Endowing \mathbb{Z}^k with the standard bilinear form $\langle e_i, e_j \rangle = \delta_{i,j}$, we get the bilinear form on \mathfrak{Q}

$$\langle \gamma_i, \gamma_j \rangle = \begin{cases} 2 & \text{if } i = j , \\ -1 & \text{if } |i - j| = 1 , \\ 0 & \text{otherwise .} \end{cases}$$

Note that this is nothing but $\langle \gamma_i, \gamma_j \rangle = c_{ij}$, where $C = (c_{ij})$ is the Cartan matrix associated with the Dinkin diagram of type A_{k-1}



with $k-1$ vertices, i.e., $C = 2 \text{ id} - A$, where A is the *adjacency matrix* of the Dinkin diagram.

Thus $\mathcal{H}_{\mathbb{F}, \mathfrak{Q}}$ can be realized as the Lie algebra over \mathbb{F} generated by \mathfrak{q}_m^i for $m \in \mathbb{Z} \setminus \{0\}$, $i = 1, \dots, k-1$ and the central element \mathfrak{c} , such that

$$\begin{cases} [\mathfrak{q}_m^i, \mathfrak{c}] = 0 & \text{for } m \in \mathbb{Z} \setminus \{0\}, i = 1, \dots, k-1 , \\ [\mathfrak{q}_m^i, \mathfrak{q}_n^j] = m \delta_{i,j} c_{ij} \mathfrak{c} & \text{for } m \in \mathbb{Z} \setminus \{0\}, i, j = 1, \dots, k-1 . \end{cases}$$

As usual we define $\mathcal{H}_{\mathbb{F}, \mathfrak{Q}}^{\pm}$, $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathfrak{Q}})$ and $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathfrak{Q}}^{\pm})$. \triangle

2.1.2. Fock space representations. Given $\mathcal{H}_{\mathbb{F}, \mathbf{L}}$ a lattice Heisenberg algebra over \mathbb{F} , we are interested in a special type of representation. Let W denote the trivial representation of $\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+$, i.e., the one dimensional \mathbb{F} -vector space with a trivial $\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+$ -action.

DEFINITION 2.5. We call *Fock space representation* of the Heisenberg algebra $\mathcal{H}_{\mathbb{F}, \mathbf{L}}$ the induced representation $\mathcal{F}_{\mathbb{F}, \mathbf{L}} := \mathcal{H}_{\mathbb{F}, \mathbf{L}} \otimes_{\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+} W$. \blacksquare

The Fock space representation is an irreducible, highest weight representation, and each element $w \in W$ is a highest weight vector, and is annihilated by $\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+$.

EXAMPLE 2.6. For the Heisenberg algebra $\mathcal{H}_{\mathbb{F}}$, the Fock space representation $\mathcal{F}_{\mathbb{F}}$ is isomorphic to the polynomial algebra $\Lambda_{\mathbb{F}} = \mathbb{F}[p_1, p_2, \dots]$ in the power sum symmetric functions (see Section 6.1.2 below for details). In this realization, the actions of the generators are given for $m \in \mathbb{Z} \setminus \{0\}$, $m > 0$, by

$$(15) \quad \mathfrak{p}_{-m} \cdot f := p_m f, \quad \mathfrak{p}_m \cdot f := m \frac{\partial f}{\partial p_m}, \quad \mathfrak{c} \cdot f := f,$$

for any $f \in \Lambda_{\mathbb{F}}$. \triangle

EXAMPLE 2.7. The Fock space representation $\mathcal{F}_{\mathbb{F}}$ of the rank k Heisenberg algebra $\mathcal{H}_{\mathbb{F}}^k$ can be realized as the tensor product of k copies of the polynomial algebra $\Lambda_{\mathbb{F}}$:

$$\mathcal{F}_{\mathbb{F}} \simeq \Lambda_{\mathbb{F}}^{\otimes k}.$$

In this realization, the action of the generators \mathfrak{p}_m^i is obvious: each copy of the Heisenberg algebra generated by \mathfrak{p}_m^i for $m \in \mathbb{Z} \setminus \{0\}$ acts on the i -th factor $\Lambda_{\mathbb{F}}$ as in Formula (15). \triangle

We conclude this section by giving the definition of Whittaker vector for Heisenberg algebras (cf. [30, Section 3]).

DEFINITION 2.8. Let $\chi: \mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+) \rightarrow \mathbb{F}$ be an algebra homomorphism such that $\chi|_{\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+} \neq 0$, and let V be a $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}})$ -module. A nonzero vector $w \in V$ is called a *Whittaker vector of type χ* if $\eta \cdot w = \chi(\eta) w$ for all $\eta \in \mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+)$. \blacksquare

REMARK 2.9. It is obvious that if V is a highest weight representation, $v \in V$ a (unique up to scalar multiple) highest weight vector, i.e., $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+)v = 0$, then two Whittaker vectors w, w' of the same type χ differ by a scalar multiple of v , as $\mathcal{U}(\mathcal{H}_{\mathbb{F}, \mathbf{L}}^+)(w - w') = 0$. \triangle

2.2. Affine Kac-Moody algebras of type \hat{A}_{k-1}

We start this section by briefly recalling the definition of the special linear algebra \mathfrak{sl}_k and the structure of its root lattice. Following [42, 62] we define the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_k$ of type \hat{A}_{k-1} by its canonical generators, and show that it can be realized as a central extension of the *loop algebra* of \mathfrak{sl}_k . In the rest of the section we give some properties of the highest weight representations of $\widehat{\mathfrak{sl}}_k$, and in particular we introduce its *basic representation*.

Let $\mathfrak{sl}_k := \mathfrak{sl}(k, \mathbb{F})$ denote the *special linear algebra* of rank $k - 1$ over \mathbb{F} . It is the Lie algebra over \mathbb{F} generated by E_i, F_i, H_i , for $i = 1, \dots, k - 1$, satisfying the following relations

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} H_j, & [H_i, H_j] &= 0, \\ [H_i, E_j] &= c_{ij} E_j, & [H_i, F_j] &= -c_{ij} F_j, \end{aligned}$$

where $C = (c_{ij})$ is the Cartan matrix of the Dynkin diagram of type A_{k-1} as in Example 2.4.

An explicit realization of the generators of \mathfrak{sl}_k inside the space of $k \times k$ complex matrices $M(k, \mathbb{C})$ is given in the following way. Let $E_{i,j}$ denote the matrix of order k with 1 in the (i, j) -entry and 0 everywhere else for $i, j = 1, \dots, k$. Define

$$E_i := E_{i,i+1} , \quad F_i := E_{i+1,i} , \quad H_i := E_{i,i} - E_{i+1,i+1} ,$$

for $i = 1, \dots, k-1$. One sees immediately that E_i, F_i, H_i satisfies the relations above.

Let us denote by \mathfrak{t} the Lie subalgebra of \mathfrak{sl}_k generated by H_i for $i = 1, \dots, k-1$ and by \mathfrak{n}_+ (resp. \mathfrak{n}_-) the Lie subalgebra of \mathfrak{sl}_k generated by E_i (resp. F_i) for $i = 1, \dots, k-1$. We have the *triangular decomposition*

$$\mathfrak{sl}_k = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+ \text{ (direct sum of vector spaces).}$$

For $i = 1, \dots, k$, define $\mathbf{e}_i \in \mathfrak{t}^*$ by

$$\mathbf{e}_i(\text{diag}(a_1, \dots, a_k)) = a_i .$$

REMARK 2.10. The elements $\gamma_i := \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 1, \dots, k-1$ form a basis of \mathfrak{t}^* . The *root lattice* \mathfrak{Q} is the lattice $\mathfrak{Q} := \oplus_{i=1}^{k-1} \mathbb{Z}\gamma_i$. We call *roots* the elements of \mathfrak{Q} , in particular the γ_i 's are the so-called *simple roots*. The lattice of positive roots is $\mathfrak{Q}_+ := \oplus_{i=1}^{k-1} \mathbb{Z}_+\gamma_i$. For a root $\gamma = \sum_{i=1}^{k-1} a_i \gamma_i \in \mathfrak{Q}$, the quantity $\text{ht}(\gamma) := \sum_{i=1}^k a_i$ is the *height* of γ . Since \mathbf{e}_i corresponds to the i -th coordinate vector in \mathbb{Z}^k , we have the following description of \mathfrak{Q} and \mathfrak{Q}_+ :

$$\begin{aligned} \mathfrak{Q} &= \{ \mathbf{e}_i - \mathbf{e}_j \in \mathbb{Z}^k \mid i, j = 1, \dots, k \} , \\ \mathfrak{Q}_+ &= \{ \mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i < j \leq k \} . \end{aligned}$$

Moreover, by setting $\langle \gamma_i, \gamma_j \rangle_{\mathfrak{Q}} := \gamma_i(H_j) = c_{ij}$, we define a nondegenerate symmetric bilinear product $\langle \cdot, \cdot \rangle_{\mathfrak{Q}}$ on \mathfrak{Q} . △

2.2.1. Definition of $\widehat{\mathfrak{sl}}_k$. Here we introduce the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_k$ of type \hat{A}_{k-1} , first via its *canonical generators* and then as a central extension of the *loop algebra* of \mathfrak{sl}_k .

DEFINITION 2.11. The *affine Kac-Moody algebra* $\widehat{\mathfrak{sl}}_k$ associated to the extended Dynkin diagram \hat{A}_{k-1} over \mathbb{F} is the Lie algebra over \mathbb{F} generated by e_i, f_i, h_i , for $i = 0, \dots, k-1$, satisfying the following relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j , & [h_i, h_j] &= 0 , \\ [h_i, e_j] &= \hat{c}_{ij} e_j , & [h_i, f_j] &= -\hat{c}_{ij} f_j , \end{aligned}$$

where $\hat{C} = (\hat{c}_{ij})$ is the Cartan matrix of the extended Dynkin diagram of type \hat{A}_{k-1} . ■

Recall that the Cartan matrix of the extended Dynkin diagram has the following form: for $k \geq 3$

$$\hat{C} = (\hat{c}_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 2 \end{pmatrix}$$

and for $k = 2$

$$\hat{C} = (\hat{c}_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Let us denote by $\hat{\mathfrak{t}}$ the Lie subalgebra of $\hat{\mathfrak{sl}}_k$ generated by h_i for $i = 0, \dots, k-1$ and by $\hat{\mathfrak{n}}_+$ (resp. $\hat{\mathfrak{n}}_-$) the Lie subalgebra of $\hat{\mathfrak{sl}}_k$ generated by e_i (resp. f_i) for $i = 0, \dots, k-1$. Then we have the *triangular decomposition*

$$(16) \quad \hat{\mathfrak{sl}}_k = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{t}} \oplus \hat{\mathfrak{n}}_+ \text{ (direct sum of vector spaces)}$$

Now we would like to describe the relation between \mathfrak{sl}_k and $\hat{\mathfrak{sl}}_k$. Define in \mathfrak{sl}_k

$$E_0 := E_{k,1}, \quad F_0 := E_{1,k}, \quad H_0 := E_{k,k} - E_{1,1}.$$

Consider now the so-called *loop algebra* $\tilde{\mathfrak{sl}}_k := \mathfrak{sl}_k \otimes \mathbb{F}[z, z^{-1}]$. Set

$$\begin{aligned} \tilde{e}_0 &:= E_0 \otimes z, & \tilde{e}_i &:= E_i \otimes 1, \\ \tilde{f}_0 &:= F_0 \otimes z^{-1}, & \tilde{f}_i &:= F_i \otimes 1 \\ \tilde{h}_0 &:= H_0 \otimes 1, & \tilde{h}_i &:= H_i \otimes 1, \end{aligned}$$

for $i = 1, \dots, k-1$.

Let us denote by \mathfrak{c} the *central element* of $\hat{\mathfrak{sl}}_k$, which is $\mathfrak{c} = \sum_{i=0}^{k-1} h_i$. So we can realize $\hat{\mathfrak{sl}}_k$ as a one-dimensional central extension

$$0 \longrightarrow \mathbb{F}\mathfrak{c} \longrightarrow \hat{\mathfrak{sl}}_k \xrightarrow{\pi} \tilde{\mathfrak{sl}}_k \longrightarrow 0,$$

where π is defined as

$$e_i \mapsto \tilde{e}_i, \quad f_i \mapsto \tilde{f}_i, \quad h_i \mapsto \tilde{h}_i,$$

for $i = 0, \dots, k-1$, and the Lie algebra structure of $\hat{\mathfrak{sl}}_k$ is obtained by

$$(17) \quad [N_1 \otimes z^m, N_2 \otimes z^n] = [N_1, N_2] \otimes z^{m+n} + m\delta_{m,-n} \text{tr}(N_1 N_2) \mathfrak{c}$$

for every $N_1, N_2 \in \mathfrak{sl}_k$ and $m, n \in \mathbb{Z}$. Thus the canonical generators of $\hat{\mathfrak{sl}}_k$ are

$$\begin{aligned} e_0 &:= E_0 \otimes z, & e_i &:= E_i \otimes 1, \\ f_0 &:= F_0 \otimes z^{-1}, & f_i &:= F_i \otimes 1, \\ h_0 &:= H_0 \otimes 1 + \mathfrak{c}, & h_i &:= H_i \otimes 1. \end{aligned}$$

Moreover, we can realize $\hat{\mathfrak{t}}$ as the one-dimensional extension

$$0 \longrightarrow \mathbb{F}\mathfrak{c} \longrightarrow \hat{\mathfrak{t}} \xrightarrow{\pi} \mathfrak{t} \longrightarrow 0.$$

REMARK 2.12. Let $\gamma_0 \in \mathfrak{sl}_k^*$ be the dual of H_0 . For $i = 1, \dots, k-1$, let \mathbf{e}_i be as in Remark 2.10. we extend \mathbf{e}_i from \mathfrak{t}^* to $\hat{\mathfrak{t}}^*$ by setting $\mathbf{e}_i(\mathfrak{c}) = 0$. Similarly, we set $\gamma_0(\mathfrak{c}) = 0$. Thus the root lattice $\hat{\mathfrak{Q}}$ of $\hat{\mathfrak{sl}}_k$ is the lattice $\oplus_{i=0}^k \mathbb{Z}\gamma_i = \mathbb{Z}\gamma_0 \oplus \mathfrak{Q}$. In a similar way, one can define the lattice of positive roots and a nondegenerate symmetric bilinear form on $\hat{\mathfrak{Q}}$. \triangle

By declaring that $\deg e_i = -\deg f_i = 1$ and $\deg h_i = 0$ for $i = 0, \dots, k-1$, we endow $\hat{\mathfrak{sl}}_k$ with a *principal \mathbb{Z} -gradation*.

$$\hat{\mathfrak{sl}}_k = \bigoplus_{i \in \mathbb{Z}} (\hat{\mathfrak{sl}}_k)_i,$$

2.2.2. Highest weight representations. Let us denote by $\mathcal{U}(\widehat{\mathfrak{sl}}_k)$ the *universal enveloping algebra* of $\widehat{\mathfrak{sl}}_k$, i.e., the unital associative algebra generated by $\widehat{\mathfrak{sl}}_k$. The principal \mathbb{Z} -gradation of $\widehat{\mathfrak{sl}}_k$ induces a \mathbb{Z} -gradation of $\mathcal{U}(\widehat{\mathfrak{sl}}_k)$:

$$\mathcal{U}(\widehat{\mathfrak{sl}}_k) = \bigoplus_{i \in \mathbb{Z}} \mathcal{U}_i .$$

Recall the triangular decomposition $\widehat{\mathfrak{sl}}_k = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{t}} \oplus \hat{\mathfrak{n}}_+$. Set $\hat{\mathfrak{b}} := \hat{\mathfrak{t}} \oplus \hat{\mathfrak{n}}_+$. Let Λ be a linear form on $\hat{\mathfrak{t}}$. We define a one-dimensional $\hat{\mathfrak{b}}$ -module $\mathbb{F}v_\Lambda$ by

$$\hat{\mathfrak{n}}_+ \cdot v_\Lambda = 0 \quad \text{and} \quad h \cdot v_\Lambda = \Lambda(h)v_\Lambda \quad \text{for any } h \in \hat{\mathfrak{t}} .$$

We consider the induced $\widehat{\mathfrak{sl}}_k$ -module

$$\tilde{V}(\Lambda) := \mathcal{U}(\widehat{\mathfrak{sl}}_k) \otimes_{\mathcal{U}(\hat{\mathfrak{b}})} \mathbb{F}v_\Lambda .$$

Setting, $\tilde{V}_i := \mathcal{U}_i v_\Lambda$, we define the principal \mathbb{Z} -gradation $\tilde{V}(\Lambda) = \bigoplus_{i \in \mathbb{Z}} \tilde{V}_i$. The $\widehat{\mathfrak{sl}}_k$ -module $\tilde{V}(\Lambda)$ contains a unique maximal proper (graded) $\widehat{\mathfrak{sl}}_k$ -submodule $I(\Lambda)$.

DEFINITION 2.13. The quotient module

$$V(\Lambda) := \tilde{V}(\Lambda)/I(\Lambda)$$

is called the *simple $\widehat{\mathfrak{sl}}_k$ -module with highest weight Λ* . The nonzero multiples of the image of v_Λ in $V(\Lambda)$ are called the *highest weight vectors* of $V(\Lambda)$. We say that Λ is *dominant* if $\Lambda(h_i) \in \mathbb{Z}_+$ for $i = 0, \dots, k-1$. ■

The \mathbb{Z} -gradation on $\tilde{V}(\Lambda)$ induces a \mathbb{Z}_+ -gradation of $V(\Lambda)$:

$$V(\Lambda) = \bigoplus_{i \in \mathbb{Z}_+} V_{-i} .$$

This gradation is called the principal gradation of $V(\Lambda)$.

DEFINITION 2.14. The *basic $\widehat{\mathfrak{sl}}_k$ -module $V(\Lambda_0)$* is the simple $\widehat{\mathfrak{sl}}_k$ -module with highest weight Λ_0 defined by

$$\Lambda(h_0) = 1 \quad \text{and} \quad \Lambda(h_i) = 0 \text{ for } i = 1, \dots, k-1 .$$
■

Define the *principal specialized character* of $V(\Lambda)$ to be

$$\text{ch}_q V(\Lambda) := \sum_{i \in \mathbb{Z}_+} (\dim V_{-i}) q^i .$$

PROPOSITION 2.15. *The principal specialized character of $V(\Lambda_0)$ is*

$$\text{ch}_q V(\Lambda_0) = \prod_{i=1}^{\infty} (1 - q^i)^{-1} .$$

2.3. Frenkel-Kac construction

Here we give a sketch of the Frenkel-Kac construction (see [42]). It is a way to induce, from a representation \mathcal{V} of the Heisenberg algebra \mathcal{H}_Ω of type A_{k-1} a representation of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_k$ of type \hat{A}_{k-1} on $\mathcal{V} \otimes \mathbb{F}[\Omega]$. We follow the presentation given in [82, Section 3.2].

By Formula (17), the subalgebra of $\widehat{\mathfrak{sl}}_k$ generated by $h_i \otimes z^m$, for $i \in \{1, \dots, k-1\}$, $m \in \mathbb{Z} \setminus \{0\}$, and \mathfrak{c} is isomorphic to the Heisenberg algebra $\mathcal{H}_{\mathbb{F}, \Omega}$. For a positive root $\gamma = \sum_{i=h}^{h'} \gamma_i \in \Omega$, define the elements of $\mathcal{H}_{\mathbb{F}, \Omega}$

$$\mathfrak{q}_m^\gamma := \mathfrak{q}_m^h \cdots \mathfrak{q}_m^{h'-1} \cdot \mathfrak{q}_m^{h'},$$

e_γ (resp. f_γ) denotes the matrix unit $E_{h, h'+1}$ (resp. $E_{h'+1, h}$) in \mathfrak{sl}_k .

Let \mathcal{V} be a representation of $\mathcal{H}_{\mathbb{F}, \Omega}$. We say that it is a *level-one* representation if \mathfrak{c} acts by the identity map. From now on let \mathcal{V} be a level-one representation of $\mathcal{H}_{\mathbb{F}, k-1}$ such that for any $v \in \mathcal{V}$ there exists an integer $m(v)$ for which

$$\mathfrak{q}_{m_1}^{l_1} \cdots \mathfrak{q}_{m_a}^{l_a} v = 0,$$

if $m_i > 0$ and $\sum_i m_i > m(v)$.

For a root $\gamma \in \Omega$, we define the generating function $X(\gamma, z)$ of operators on $\mathcal{V} \otimes \mathbb{F}[\Omega]$ by

$$X(\gamma, z) = \exp \left(\sum_{m=1}^{\infty} \frac{z^m}{m} \mathfrak{q}_{-m}^\gamma \right) \exp \left(- \sum_{m=1}^{\infty} \frac{z^{-m}}{m} \mathfrak{q}_m^\gamma \right) \exp(\log z \cdot \mathfrak{c} + \gamma),$$

where $\exp(\log z \cdot \mathfrak{c} + \gamma)$ is the operator defined by

$$\exp(\log z \cdot \mathfrak{c} + \gamma)(v \otimes [\beta]) := z^{\frac{1}{2}\langle \gamma, \gamma \rangle_\Omega + \langle \gamma, \beta \rangle_\Omega} (v \otimes [\gamma + \beta])$$

for $v \otimes [\beta] \in \mathcal{V} \otimes \mathbb{F}[\Omega]$.

Let $X_m(\gamma)$ denote the operator given by $X(\gamma, z) = \sum_{m \in \mathbb{Z}} X_m(\gamma) z^m$. We define a map $\epsilon: \Omega \times \Omega \rightarrow \{\pm 1\}$ by

$$\epsilon(\gamma_i, \gamma_j) = \begin{cases} -1 & j = i, i+1, \\ 1 & \text{otherwise,} \end{cases}$$

with the conditions $\epsilon(\gamma + \gamma', \beta) = \epsilon(\gamma, \beta)\epsilon(\gamma', \beta)$ and $\epsilon(\gamma, \beta + \beta') = \epsilon(\gamma, \beta)\epsilon(\gamma, \beta')$.

THEOREM 2.16. [42, Theorem 1] *The vector space $\mathcal{V} \otimes \mathbb{F}[\Omega]$ has an $\widehat{\mathfrak{sl}}_k$ -module structure given by*

$$\begin{aligned} (h_i \otimes 1)(v \otimes [\beta]) &= \langle \gamma_i, \beta \rangle_\Omega (v \otimes [\beta]), \\ (h_i \otimes z^m)(v \otimes [\beta]) &= \mathfrak{q}_m^i v \otimes [\beta], \\ (e_\gamma \otimes z^m)(v \otimes [\beta]) &= \epsilon(\gamma, \beta) X_m(\gamma)(v \otimes [\beta]), \\ (f_\gamma \otimes z^m)(v \otimes [\beta]) &= \epsilon(\beta, \gamma) X_{-m}(-\gamma)(v \otimes [\beta]), \end{aligned}$$

and $c = 1$, i.e., it is a level 1 representation. Moreover, if \mathcal{V} was an irreducible (highest weight) representation of \mathcal{H}_Ω , $\mathcal{V} \otimes \mathbb{F}[\Omega]$ is an irreducible (highest weight) representation of $\widehat{\mathfrak{sl}}_k$. In particular if \mathcal{V} is equivalent to the Fock space of \mathcal{H}_Ω , then $\mathcal{V} \otimes \mathbb{F}[\Omega]$ is equivalent to the basic representation of $\widehat{\mathfrak{sl}}_k$.

CHAPTER 3

Moduli of framed sheaves on projective stacks

In this chapter we introduce the theory of $((\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves and their moduli spaces. Our treatment is based on a forthcoming paper by Bruzzo and Sala [23]. In Section 3.1 we introduce semistability conditions for framed sheaves on projective stacks, and study boundedness of families of such objects. In Section 3.2 we present the construction of moduli stacks and moduli spaces of framed sheaves on projective stacks via GIT. In Section 3.3 we restrict ourselves to two-dimensional projective toric orbifolds and study the case of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves, i.e., sheaves that on a divisor \mathcal{D} are framed to a locally free sheaf $\mathcal{F}_{\mathcal{D}}$. Finally, in Section 3.4 we apply the theory to the case of toric orbifolds, in particular to root stack compactifications of a smooth open toric surface.

3.1. Framed sheaves on projective stacks

In this section, following [23, Section 3], we give some elements of the theory of δ -(semi)stable framed sheaves on projective stacks. Most results are rather straightforward generalizations from the case of smooth projective varieties [57, 56]. We refer to those papers as the main references for framed sheaves on schemes.

3.1.1. Preliminaries. Let \mathcal{X} be a projective stack of dimension d with coarse moduli scheme $\mathcal{X} \xrightarrow{\pi} X$. Let $(\mathcal{G}, \mathcal{O}_X(1))$ be a polarization on \mathcal{X} . Fix a coherent sheaf \mathcal{F} on \mathcal{X} and a polynomial

$$\delta(n) := \delta_1 \frac{n^{d-1}}{(d-1)!} + \delta_2 \frac{n^{d-2}}{(d-2)!} + \cdots + \delta_d \in \mathbb{Q}[n]$$

with $\delta_1 > 0$. We call \mathcal{F} a *framing sheaf* and δ a *stability polynomial*.

DEFINITION 3.1. A *framed sheaf* on \mathcal{X} is a pair $\mathfrak{E} := (\mathcal{E}, \phi_{\mathcal{E}})$, where \mathcal{E} is a coherent sheaf on \mathcal{X} and $\phi_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{F}$ is a morphism of sheaves. We call $\phi_{\mathcal{E}}$ a *framing* of \mathcal{E} . ■

First note that the pair $F_{\mathcal{G}}(\mathfrak{E}) := (F_{\mathcal{G}}(\mathcal{E}), F_{\mathcal{G}}(\phi_{\mathcal{E}}): F_{\mathcal{G}}(\mathcal{E}) \rightarrow F_{\mathcal{G}}(\mathcal{F}))$ is a framed sheaf on X . Moreover, since $F_{\mathcal{G}}$ is an exact functor (cf. Remark 1.2), we have $\ker(F_{\mathcal{G}}(\phi_{\mathcal{E}})) = F_{\mathcal{G}}(\ker(\phi_{\mathcal{E}}))$ and $\text{Im}(F_{\mathcal{G}}(\phi_{\mathcal{E}})) = F_{\mathcal{G}}(\text{Im}(\phi_{\mathcal{E}}))$. Therefore by Lemma 1.21, $F_{\mathcal{G}}(\phi_{\mathcal{E}})$ is zero if and only if $\phi_{\mathcal{E}}$ is zero.

For any framed sheaf $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$, its dimension, Hilbert polynomial, multiplicity, \mathcal{G} -rank and hat-slope are just the corresponding quantities for its underlying coherent sheaf \mathcal{E} .

Define the function $\epsilon(\phi_{\mathcal{E}})$ by

$$\epsilon(\phi_{\mathcal{E}}) := \begin{cases} 1 & \text{if } \phi_{\mathcal{E}} \neq 0, \\ 0 & \text{if } \phi_{\mathcal{E}} = 0. \end{cases}$$

The *framed Hilbert polynomial* of \mathfrak{E} is

$$P_{\mathcal{G}}(\mathfrak{E}, n) := P_{\mathcal{G}}(\mathcal{E}, n) - \epsilon(\phi_{\mathcal{E}})\delta(n) ,$$

and its *reduced framed Hilbert polynomial* is

$$p_{\mathcal{G}}(\mathfrak{E}, n) := \frac{P_{\mathcal{G}}(\mathfrak{E}, n)}{\alpha_{\mathcal{G}, \dim(\mathfrak{E})}(\mathcal{E})} .$$

The *framed hat-slope* of $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ is

$$\hat{\mu}_{\mathcal{G}}(\mathfrak{E}) := \hat{\mu}_{\mathcal{G}}(\mathcal{E}) - \frac{\epsilon(\phi_{\mathcal{E}})\delta_1}{\alpha_{\mathcal{G}, \dim(\mathfrak{E})}(\mathcal{E})} .$$

If \mathcal{E}' is a subsheaf of \mathcal{E} with quotient $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$, the framing $\phi_{\mathcal{E}}$ induces framings $\phi_{\mathcal{E}'} := \phi_{\mathcal{E}|_{\mathcal{E}'}}$ on \mathcal{E}' and $\phi_{\mathcal{E}''}$ on \mathcal{E}'' , where the framing $\phi_{\mathcal{E}''}$ is defined as $\phi_{\mathcal{E}''} = 0$ if $\phi_{\mathcal{E}'} \neq 0$; otherwise, $\phi_{\mathcal{E}''}$ is the induced morphism on \mathcal{E}'' . If $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ is a framed sheaf on \mathcal{X} and \mathcal{E}' is a subsheaf of \mathcal{E} , we denote by \mathfrak{E}' the framed sheaf $(\mathcal{E}', \phi_{\mathcal{E}'})$ and by \mathfrak{E}'' the framed sheaf $(\mathcal{E}'', \phi_{\mathcal{E}''})$. With this convention the framed Hilbert polynomial of \mathfrak{E} behaves additively:

$$P_{\mathcal{G}}(\mathfrak{E}) = P_{\mathcal{G}}(\mathfrak{E}') + P_{\mathcal{G}}(\mathfrak{E}'') .$$

The same property holds for the framed hat-slope.

DEFINITION 3.2. A *morphism of framed sheaves* $f: \mathfrak{E} \rightarrow \mathfrak{H}$ is a morphism of the underlying coherent sheaves $f: \mathcal{E} \rightarrow \mathcal{H}$ for which there is an element $\lambda \in k$ such that $\phi_{\mathcal{H}} \circ f = \lambda \phi_{\mathcal{E}}$. We say that f is injective (resp. surjective) if the morphism $f: \mathcal{E} \rightarrow \mathcal{H}$ is injective (resp. surjective). If f is injective, we call \mathfrak{E} a *framed submodule* of \mathfrak{H} . If f is surjective, we call \mathfrak{H} a *framed quotient module* of \mathfrak{E} . \blacksquare

LEMMA 3.3. [56, Lemma 1.5] *The set $\text{Hom}(\mathfrak{E}, \mathfrak{H})$ of morphisms of framed sheaves is a linear subspace of $\text{Hom}(\mathcal{E}, \mathcal{H})$. If $f: \mathfrak{E} \rightarrow \mathfrak{H}$ is an isomorphism, the factor λ can be taken in k^* . In particular, the isomorphism $f_0 = \lambda^{-1}f$ satisfies $\phi_{\mathcal{H}} \circ f_0 = \phi_{\mathcal{E}}$.*

3.1.2. Semistability. We use the following convention: if the word “(semi)stable” occurs in any statement in combination with the symbol (\leq) , two variants of the statement are understood at the same time: a “semistable” one involving the relation “ \leq ” and a “stable” one involving the relation “ $<$ ”.

We give a definition of δ -(semi)stability for d -dimensional framed sheaves.

DEFINITION 3.4. A d -dimensional framed sheaf $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ is said to be δ -(semi)stable if and only if the following conditions are satisfied:

- (i) $P_{\mathcal{G}}(\mathcal{E}') (\leq) \alpha_{\mathcal{G}, d}(\mathcal{E}') p_{\mathcal{G}}(\mathfrak{E})$ for all subsheaves $\mathcal{E}' \subseteq \ker \phi_{\mathcal{E}}$,
- (ii) $(P_{\mathcal{G}}(\mathcal{E}') - \delta) (\leq) \alpha_{\mathcal{G}, d}(\mathcal{E}') p_{\mathcal{G}}(\mathfrak{E})$ for all subsheaves $\mathcal{E}' \subset \mathcal{E}$.

\blacksquare

By using the same arguments as in the proof of Lemma 1.2 in [56], one can prove the following.

LEMMA 3.5. [23, Lemma 3.7] *Let $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ be a d -dimensional framed sheaf. If \mathfrak{E} is δ -semistable, then $\ker \phi_{\mathcal{E}}$ is torsion-free.*

DEFINITION 3.6. Let $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ be a framed sheaf with $\alpha_{\mathcal{G},d}(\mathcal{E}) = 0$. If $\phi_{\mathcal{E}}$ is injective, we say that \mathfrak{E} is *semistable* (in this case, the definition of semistability of the corresponding framed sheaves does not depend on δ). Moreover, if $P_{\mathcal{G}}(\mathcal{E}) = \delta$ we say that \mathfrak{E} is δ -stable. ■

We conclude this section with the definition of Jordan-Hölder filtrations. The construction does not differ from the case of framed sheaves on smooth projective varieties, and the existence in the case of projective stacks is granted by the fact that $F_{\mathcal{G}}$ is an exact functor and is *compatible* with the torsion filtration (cf. Corollary 1.24).

DEFINITION 3.7. Let $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ be a δ -semistable d -dimensional framed sheaf. A *Jordan-Hölder filtration* of \mathfrak{E} is a filtration

$$\mathcal{E}_{\bullet} : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E} ,$$

such that all the factors $\mathcal{E}_i/\mathcal{E}_{i-1}$ together with the induced framings ϕ_i are δ -stable with framed Hilbert polynomial $P_{\mathcal{G}}(\mathcal{E}_i/\mathcal{E}_{i-1}, \phi_i) = \alpha_{\mathcal{G},d}(\mathcal{E}_i/\mathcal{E}_{i-1})p_{\mathcal{G}}(\mathfrak{E})$. ■

A straightforward generalization of [56, Proposition 1.13], yields the following result.

PROPOSITION 3.8. [23, Proposition 3.14] *Every δ -semistable framed sheaf \mathfrak{E} admits a Jordan-Hölder filtration. The framed sheaf*

$$gr(\mathfrak{E}) = (gr(\mathcal{E}), gr(\phi_{\mathcal{E}})) := \bigoplus_i (\mathcal{E}_i/\mathcal{E}_{i-1}, \phi_i)$$

does not depend, up to isomorphism, on the choice of the Jordan-Hölder filtration.

DEFINITION 3.9. Two δ -semistable framed sheaves $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ and $\mathfrak{H} = (\mathcal{H}, \phi_{\mathcal{H}})$ with the same reduced framed Hilbert polynomial are called *S-equivalent* if their associated graded objects $gr(\mathfrak{E})$ and $gr(\mathfrak{H})$ are isomorphic. ■

3.1.3. Families of framed sheaves. We introduce the notion of family of framed sheaves on a projective stack \mathcal{X} , parametrized by a base scheme S .

DEFINITION 3.10. A *flat family* $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ of framed sheaves on \mathcal{X} parameterized by a scheme S consists of a coherent sheaf \mathcal{E} on $\mathcal{X} \times S$, flat over S , a line bundle $L_{\mathcal{E}}$ on S , and a morphism $\phi_{\mathcal{E}} : L_{\mathcal{E}} \rightarrow p_{S*} \mathcal{H}om(\mathcal{E}, p_{\mathcal{X}}^* \mathcal{F})$ called a *framing* of \mathcal{E} , where \mathcal{F} is the framing sheaf. Two families $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ and $\mathfrak{E}' = (\mathcal{E}', L_{\mathcal{E}'}, \phi_{\mathcal{E}'})$ are isomorphic if there exist isomorphisms $g : \mathcal{E} \rightarrow \mathcal{E}'$ and $h : L_{\mathcal{E}} \rightarrow L_{\mathcal{E}'}$ such that

$$(18) \quad \tilde{g} \circ \phi_{\mathcal{E}} = \phi_{\mathcal{E}'} \circ h ,$$

where

$$\tilde{g} : p_{S*} \mathcal{H}om(\mathcal{E}, p_{\mathcal{X}}^* \mathcal{F}) \rightarrow p_{S*} \mathcal{H}om(\mathcal{E}', p_{\mathcal{X}}^* \mathcal{F})$$

is the isomorphism induced by g . ■

REMARK 3.11. We may look at a framing $\phi_{\mathcal{E}} : L_{\mathcal{E}} \rightarrow p_{S*} \mathcal{H}om(\mathcal{E}, p_{\mathcal{X}}^* \mathcal{F})$ as a nowhere vanishing morphism

$$\tilde{\phi}_{\mathcal{E}} : p_S^* L_{\mathcal{E}} \otimes \mathcal{E} \rightarrow p_{\mathcal{X}}^* \mathcal{F} ,$$

defined as the composition

$$p_S^* L_{\mathcal{E}} \otimes \mathcal{E} \rightarrow p_S^* p_{S*} \mathcal{H}om(\mathcal{E}, p_{\mathcal{X}}^* \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{H}om(\mathcal{E}, p_{\mathcal{X}}^* \mathcal{F}) \otimes \mathcal{E} \xrightarrow{\text{ev}} p_{\mathcal{X}}^* \mathcal{F} .$$

△

We say that the flat family $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ has the property **P** if for any closed point $s \in S$ the framed sheaf $(\mathcal{E}_s, (\tilde{\phi}_{\mathcal{E}})_s: p_s^*((L_{\mathcal{E}})_s) \otimes \mathcal{E}_s \rightarrow p_{\mathcal{X}}^*(\mathcal{F})_s)$ has the property **P**, where $p_s: \text{Spec}(k(s)) \times \mathcal{X} \rightarrow \text{Spec}(k(s))$ is the projection.

DEFINITION 3.12. Let $\mathfrak{H} = (\mathcal{H}, L_{\mathcal{H}}, \phi_{\mathcal{H}})$ be a flat family of framed sheaves on \mathcal{X} parameterized by S . A *flat family of framed quotients* of \mathfrak{H} is a flat family of framed sheaves $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ on \mathcal{X} parameterized by S with an epimorphism $q: \mathcal{H} \rightarrow \mathcal{E}$ and a morphism $\sigma \in \text{Hom}(L_{\mathcal{E}}, L_{\mathcal{H}})$ such that the diagram

$$\begin{array}{ccc}
 p_S^* L_{\mathcal{E}} \otimes \mathcal{H} & \xrightarrow{\text{id}_{p_S^* L_{\mathcal{E}}} \otimes q} & p_S^* L_{\mathcal{E}} \otimes \mathcal{E} \\
 \searrow p_S^* \sigma \otimes \text{id}_{\mathcal{H}} & & \downarrow \tilde{\phi}_{\mathcal{E}} \\
 & p_S^* L_{\mathcal{H}} \otimes \mathcal{H} & \\
 & \searrow \tilde{\phi}_{\mathcal{H}} & \\
 & & p_{\mathcal{X}}^* \mathcal{F}
 \end{array}$$

commutes. ■

REMARK 3.13. Let $\mathfrak{H} = (\mathcal{H}, \phi_{\mathcal{H}})$ be a framed sheaf on \mathcal{X} . Given a scheme S , by pulling \mathfrak{H} back to $\mathcal{X} \times S$ one defines a flat family $(p_{\mathcal{X}}^*(\mathcal{H}), \mathcal{O}_S, p_{\mathcal{X}}^*(\phi_{\mathcal{H}}))$ parameterized by S . A flat family of framed quotients of \mathfrak{H} is a flat family of framed sheaves $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ on \mathcal{X} parameterized by S with an epimorphism $q: p_{\mathcal{X}}^*(\mathcal{H}) \rightarrow \mathcal{E}$ and a section $\sigma \in \Gamma(S, L_{\mathcal{E}}^{\vee})$ such that the previous diagram commutes. △

3.2. Construction of the moduli spaces

In this section we describe the construction of the moduli spaces of δ -(semi)stable framed sheaves on a normal irreducible projective stack \mathcal{X} , as it was stated in [23, Section 4]. If the framing vanishes, these are just the moduli spaces of (semi)stable torsion-free sheaves, for which we refer to Nironi's paper [91]. From now on we shall always assume that the framings are nonzero unless the contrary is explicitly stated.

Let \mathcal{X} be a d -dimensional projective stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$. In this section we make the following assumptions on \mathcal{X} :

- \mathcal{X} is normal, which is used in the proof Proposition 3.16 (we do not want to go into detail here, we only say that the normality hypothesis is necessary to deal with framed sheaves with non torsion-free kernels);
- \mathcal{X} is irreducible. By [107, Lemma 2.3], also the coarse moduli scheme X is irreducible. We shall use this hypothesis in the proof of Proposition 3.26, which is in turn used to prove that the moduli space of δ -stable framed sheaves is fine.

3.2.1. GIT. The construction of the moduli spaces of δ -(semi)stable framed sheaves on \mathcal{X} is quite involved, hence, for the sake of clarity, we divide it into several steps.

Step 1: construction of a “Quot-like” scheme that also takes the framing into account. By [91], Proposition 4.20, the functor $F_{\mathcal{G}}$ defines a closed embedding of $\text{Quot}_{\mathcal{X}/k}(\mathcal{E}, P_0)$ into $\text{Quot}_{X/k}(F_{\mathcal{G}}(\mathcal{E}), P_0)$, for any coherent sheaf \mathcal{E} on \mathcal{X} and numerical polynomial P_0 of degree d . In particular, $\text{Quot}_{\mathcal{X}/k}(\mathcal{E}, P_0)$ is a projective scheme.

Let P_0 denote a numerical polynomial of degree d , $P = P_0 - \delta$. Fix an integer m sufficiently large and let V be a vector space of dimension $P_0(m)$. For every sheaf E on X we shall denote $E(-m) = E \otimes \mathcal{O}_X(-m)$.

Set $\tilde{\mathbf{Q}} := \text{Quot}_{\mathcal{X}/k}(G_{\mathcal{G}}(V(-m)), P_0)$ and $\mathbf{P} := \mathbb{P}(\text{Hom}(V, H^0(\mathcal{F} \otimes \mathcal{G}^\vee \otimes \pi^* \mathcal{O}_X(m))))^\vee \simeq \mathbb{P}(\text{Hom}(V, H^0(F_{\mathcal{G}}(\mathcal{F})(m))))^\vee$. Given a point $[a: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{F})(m))]$ in \mathbf{P} we can define a framing on $G_{\mathcal{G}}(V(-m))$ as follows. Let us consider the composition

$$V(-m) \xrightarrow{\text{aoid}} H^0(F_{\mathcal{G}}(\mathcal{F})(m))(-m) \xrightarrow{\text{ev}} F_{\mathcal{G}}(\mathcal{F}) .$$

By applying the functor $G_{\mathcal{G}}$ and composing on the right with $\theta_{\mathcal{G}}(\mathcal{F})$, we obtain

$$\phi_a: G_{\mathcal{G}}(V(-m)) \xrightarrow{G_{\mathcal{G}}(\text{aoid})} H^0(F_{\mathcal{G}}(\mathcal{F})(m)) \otimes G_{\mathcal{G}}(\mathcal{O}_X(-m)) \xrightarrow{G_{\mathcal{G}}(\text{ev})} G_{\mathcal{G}}(F_{\mathcal{G}}(\mathcal{F})) \xrightarrow{\theta_{\mathcal{G}}(\mathcal{F})} \mathcal{F} .$$

Let $i: Z' \hookrightarrow \tilde{\mathbf{Q}} \times \mathbf{P}$ be the closed subscheme of points

$$([\tilde{q}: G_{\mathcal{G}}(V(-m)) \rightarrow \mathcal{E}], [a: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{F})(m))])$$

such that the framing ϕ_a factors through \tilde{q} and induces a framing $\phi_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{F}$.

We explain how to define a flat family of framed sheaves on \mathcal{X} parameterized by $Z' \subset \tilde{\mathbf{Q}} \times \mathbf{P}$. Let $\tilde{\mathbf{q}}: p_{\tilde{\mathbf{Q}} \times \mathcal{X}, \mathcal{X}}^* G_{\mathcal{G}}(V(-m)) \rightarrow \tilde{\mathcal{U}}$ be the universal quotient family on \mathcal{X} parameterized by $\tilde{\mathbf{Q}}$. Set

$$\mathcal{H} := \left(p_{\tilde{\mathbf{Q}} \times \mathcal{X}, \mathcal{X}} \circ p_{\tilde{\mathbf{Q}} \times \mathbf{P} \times \mathcal{X}, \tilde{\mathbf{Q}} \times \mathcal{X}} \right)^* G_{\mathcal{G}}(V(-m)) .$$

Then we have a quotient morphism

$$p_{\tilde{\mathbf{Q}} \times \mathbf{P} \times \mathcal{X}, \tilde{\mathbf{Q}} \times \mathcal{X}}^* \tilde{\mathbf{q}}: \mathcal{H} \rightarrow p_{\tilde{\mathbf{Q}} \times \mathbf{P} \times \mathcal{X}, \tilde{\mathbf{Q}} \times \mathcal{X}}^* \tilde{\mathcal{U}} \rightarrow 0 .$$

Consider now the universal quotient sheaf of \mathbf{P} , that is,

$$\rho: \text{Hom}(V \otimes \mathcal{O}_{\mathbf{P}}, H^0(F_{\mathcal{G}}(\mathcal{F})(m)) \otimes \mathcal{O}_{\mathbf{P}}) \rightarrow \mathcal{O}_{\mathbf{P}}(1) \rightarrow 0 .$$

By an argument similar to the one used earlier to construct ϕ_a from a point $[a] \in \mathbf{P}$, we can define a morphism

$$\phi_{\mathcal{H}}: L_{\mathcal{H}} \rightarrow p_{\tilde{\mathbf{Q}} \times \mathbf{P} \times \mathcal{X}, \tilde{\mathbf{Q}} \times \mathbf{P}}^* \text{Hom}(\mathcal{H}, p_{\tilde{\mathbf{Q}} \times \mathbf{P} \times \mathcal{X}, \mathcal{X}}^* \mathcal{F}) ,$$

where $L_{\mathcal{H}} := p_{\tilde{\mathbf{Q}} \times \mathbf{P}, \mathbf{P}}^* \mathcal{O}_{\mathbf{P}}(-1)$. In this way, $(\mathcal{H}, L_{\mathcal{H}}, \phi_{\mathcal{H}})$ is a flat family of framed sheaves on the stack \mathcal{X} parameterized by $\tilde{\mathbf{Q}} \times \mathbf{P}$.

We can endow the universal quotient family $\mathcal{U} := (i \times \text{id}_{\mathcal{X}})^* \tilde{\mathcal{U}}$ on \mathcal{X} parameterized by Z' with a framed sheaf structure in the following way. By the definition of Z' there exists a morphism

$$\phi_{\mathcal{U}}: L_{\mathcal{U}} \rightarrow p_{Z' \times \mathcal{X}, Z'}^* \text{Hom}(\mathcal{U}, p_{Z' \times \mathcal{X}, \mathcal{X}}^* \mathcal{F}) ,$$

where $L_{\mathcal{U}} := \left(p_{\tilde{\mathbf{Q}} \times \mathbf{P}, \mathbf{P}} \circ i \right)^* \mathcal{O}_{\mathbf{P}}(-1) = i^* L_{\mathcal{H}}$.

Set $\mathcal{U} := (\mathcal{U}, L_{\mathcal{U}}, \phi_{\mathcal{U}})$. Then by choosing the morphism σ in Definition 3.12 to be $\text{id}_{L_{\mathcal{U}}}$, we obtain the following result.

PROPOSITION 3.14. [23, Proposition 4.1] \mathfrak{U} is a flat family of framed sheaves on \mathcal{X} parameterized by Z' , and is formed by framed quotients of the flat family $\mathfrak{H} := (i^*\mathcal{H}, i^*L_{\mathcal{H}}, i^*\phi_{\mathcal{H}})$ of framed sheaves on \mathcal{X} , which is also parameterized by Z' .

The schemes $\tilde{\mathbf{Q}}$ and \mathbf{P} enjoy universality properties so that the same happens for the scheme Z' . This is proved as in [21] and [56].

PROPOSITION 3.15. [23, Proposition 4.2] Let $[a]$ be a point in \mathbf{P} , and let $\mathfrak{E} = (\mathcal{E}, L_{\mathcal{E}}, \phi_{\mathcal{E}})$ be a flat family of framed quotients of $(G_{\mathcal{G}}(V(-m)), \phi_a)$. Assume that the Hilbert polynomial of \mathcal{E}_s is independent of $s \in S$. There is a morphism $f: S \rightarrow Z'$ (unique up to a unique isomorphism) such that \mathfrak{E} is isomorphic to the pull-back of \mathfrak{U} via $f \times \text{id}$.

Step 2: $\text{GL}(V)$ -action on Z' . Until now, we constructed a projective scheme Z' which parameterizes a flat family of framed quotients of $G_{\mathcal{G}}(V(-m))$, with its framed sheaf structure. To use the GIT machinery we need to define an action of a reductive group on Z' . We shall endow Z' of a $\text{GL}(V)$ -action induced by $\text{GL}(V)$ -actions on $\tilde{\mathbf{Q}}$ and \mathbf{P} . The action is formally given in [23, Section 4], here we just describe the action pointwise.

The right $\text{GL}(V)$ -action on $\tilde{\mathbf{Q}}$ is pointwise defined by

$$[\tilde{q}] \cdot g := [\tilde{q} \circ (g \otimes \text{id})]$$

where $[\tilde{q}: G_{\mathcal{G}}(V(-m)) \rightarrow \mathcal{E}]$ is a closed point in $\tilde{\mathbf{Q}}$ and $g \in \text{GL}(V)$. The right action of $\text{GL}(V)$ on \mathbf{P} is given by

$$[a] \circ g := [a \circ g]$$

for any closed point $[a: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{F})(m))]$ and $g \in \text{GL}(V)$.

Step 3: comparison between GIT (semi)stability and the δ -(semi)stability condition for framed sheaves. We need to define suitable $\text{SL}(V)$ -linearized ample line bundles on Z' which will allow us to deal with GIT (semi)stable points on Z' and compare them to δ -(semi)stable framed sheaves on \mathcal{X} . From now on we consider $\text{SL}(V)$ instead of $\text{GL}(V)$ because the study of the GIT (semi)stable points is easier for the first group.

As it is described in [91, Section 6.1], one can define line bundles on $\tilde{\mathbf{Q}}$

$$L_l := \det(p_{\tilde{\mathbf{Q}}}^* F_{\mathcal{G}}(\tilde{\mathcal{U}})(l)) .$$

By [91, Proposition 6.2], for l sufficiently large the line bundles L_l are very ample. Moreover, they carry natural $\text{SL}(V)$ -linearizations (cf. [91, Lemma 6.3]). Then the ample line bundles

$$\mathcal{O}_{Z'}(n_1, n_2) := q_{\tilde{\mathbf{Q}}}^* L_l^{\otimes n_1} \otimes q_{\mathbf{P}}^* \mathcal{O}_{\mathbf{P}}(n_2)$$

carry natural $\text{SL}(V)$ -linearizations, where $q_{\tilde{\mathbf{Q}}}$ and $q_{\mathbf{P}}$ are the natural projections from Z' to $\tilde{\mathbf{Q}}$ and \mathbf{P} respectively. As explained in [56, Section 3], only the ratio n_2/n_1 matters, and we choose it to be

$$\frac{n_2}{n_1} := P(l) \frac{\delta(m)}{P(m)} - \delta(l) ,$$

assuming, of course, that l is chosen large enough to make this term positive.

To use the GIT machinery we need to compare the GIT (semi)stability with the δ -(semi)stability condition for framed sheaves. The results we show in the following are generalizations of those proved in [56, Section 3] for framed sheaves on smooth projective varieties. The proofs are rather straightforward due to the properties of the functors $F_{\mathcal{G}}$ and $G_{\mathcal{G}}$.

Using the stacky version of the Grothendieck Lemma ([91, Lemma 4.13]) and the projectivity of the Quot scheme for coherent sheaves on stacks [93], one can prove that torsion-freeness is an open property. Thus there is an open subscheme $U \subset Z'$ whose points represent framed sheaves with torsion-free kernel. We assume that U is nonempty and denote by Z its closure in Z' .

Let $\tilde{q}: G_{\mathcal{G}}(V(-m)) \rightarrow \mathcal{E}$ be a morphism representing a point $[\tilde{q}] \in \tilde{\mathbf{Q}}$. By applying the functor $F_{\mathcal{G}}$ to \tilde{q} and then composing on the left by $\varphi_{\mathcal{G}}(V(-m))$, we obtain

$$V(-m) \xrightarrow{\varphi_{\mathcal{G}}(V(-m))} F_{\mathcal{G}}(G_{\mathcal{G}}(V(-m))) \rightarrow F_{\mathcal{G}}(\mathcal{E}),$$

and in cohomology we get $q: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{E})(m))$. By combining the arguments in [56, Proposition 3.2], with those in [91, Theorem 5.1] we obtain the following result.

PROPOSITION 3.16. [23, Proposition 4.4] *For sufficiently large l , a point $([\tilde{q}], [a]) \in Z$ is (semi)stable with respect to the $\mathrm{SL}(V)$ -action on Z if and only if the corresponding framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ is δ -(semi)stable and the map $q: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{E})(m))$ induced by \tilde{q} .*

By using similar arguments as in [58, Lemma 4.3.2], we can prove the following.

LEMMA 3.17. [23, Lemma 4.5] *Let $([\tilde{q}: G_{\mathcal{G}}(V(-m)) \rightarrow \mathcal{E}], [a: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{F})(m))])$ be a closed point of Z' such that $F_{\mathcal{G}}(\mathcal{E})(m)$ is globally generated and the map $q: V \rightarrow H^0(F_{\mathcal{G}}(\mathcal{E})(m))$ induced by \tilde{q} is an isomorphism. There is a natural injective homomorphism $i: \mathrm{Aut}(\mathcal{E}, \phi_{\mathcal{E}}) \rightarrow \mathrm{GL}(V)$ whose image is precisely the stabilizer subgroup $\mathrm{GL}(V)_{([\tilde{q}], [a])}$ of the point $([\tilde{q}], [a])$.*

Step 4: Good and geometric quotients and (semi)stable locus. Thanks to the results we proved before, we are ready to use [58, Theorem 4.2.10], which allows us to construct a (quasi-)projective scheme *parameterizing* (semi)stable points of Z . In order to do this, we first recall the notions of *good* and *geometric* quotients.

DEFINITION 3.18. [58, Definition 4.2.2] Let G be an affine algebraic group acting on a scheme Y . A morphism $f: Y \rightarrow W$ is a *good quotient* if

- f is affine and G -invariant,
- f is surjective, and $U \subset W$ is open if and only if $f^{-1}(U) \subset Y$ is open,
- the natural morphism $\mathcal{O}_W \rightarrow f_*(\mathcal{O}_Y)$ is an isomorphism,
- If V is an invariant closed subset of Y , then $f(V)$ is a closed subset of W . If V_1 and V_2 are disjoint invariant closed subsets of Y , then $f(V_1) \cap f(V_2) = \emptyset$.

The morphism f is said to be a *geometric quotient* if it is a good quotient and the geometric fibers of f are the orbits of the geometric points of Y . Finally, f is a *universal good (geometric) quotient* if $W' \times_W Y \rightarrow W'$ is a good (geometric) quotient for any morphism $W' \rightarrow W$ of k -schemes. ■

A (universal) good quotient is in particular a (universal) categorical quotient, i.e., if $f: Y \rightarrow W$ is a (universal) good quotient and $g: Y \rightarrow T$ is a G -invariant morphism, then there exists a unique $h: W \rightarrow T$ such that $g = h \circ f$.

Let $Z^s \subset Z^{ss} \subset Z$ denote the open subschemes of stable and semistable points of Z , respectively. By Proposition 3.16 a point in $Z^{(s)s}$ corresponds — roughly speaking — to a δ -(semi)stable framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ on \mathcal{X} together with the choice of a basis in $H^0(F_{\mathcal{G}}(\mathcal{E})(m))$.

We denote by $\mathfrak{U}^{(s)s} = (\mathcal{U}^{(s)s}, L_{\mathcal{U}^{(s)s}}, \phi_{\mathcal{U}^{(s)s}})$ the universal family of δ -(semi)stable framed sheaves on \mathcal{X} parameterized by $Z^{(s)s}$ induced, through pull-back, by the one parameterized by Z' .

By using [58, Theorem 4.2.10], we get directly the following.

THEOREM 3.19. [23, Theorem 4.7] *There exists a projective scheme*

$$M^{ss} = M_{\mathcal{X}/k}^{ss}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta)$$

and a morphism $\tilde{\pi}: Z^{ss} \rightarrow M^{ss}$ such that $\tilde{\pi}$ is a universal good quotient for the $SL(V)$ -action on Z^{ss} . Moreover, there is an open subscheme

$$M^s = M_{\mathcal{X}/k}^s(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta) \subset M^{ss}$$

such that $Z^s = \tilde{\pi}^{-1}(M^s)$ and $\tilde{\pi}: Z^s \rightarrow M^s$ is a universal geometric quotient. Finally, there is a positive integer l and a very ample line bundle $\mathcal{O}_{M^{ss}}(1)$ on M^{ss} such that $\mathcal{O}_{Z'}(n_1, n_2)|_{Z^{ss}}^{\otimes l} \simeq \tilde{\pi}^(\mathcal{O}_{M^{ss}}(1))$.*

By using the same arguments as in the proof of [56, Proposition 3.3], and the semicontinuity theorem for Hom groups of flat families of framed sheaves [23, Appendix A], we get the following result.

PROPOSITION 3.20. [23, Proposition 4.8] *Two points $([\tilde{q}], [a])$ and $([\tilde{q}'], [a'])$ in Z^{ss} are mapped to the same point in M^{ss} if and only if the corresponding framed sheaves are S -equivalent.*

3.2.2. The moduli stacks of δ -(semi)stable framed sheaves. In the previous section we used GIT machinery to construct a good (geometric) quotient $M^{(s)s}$ of $Z^{(s)s}$. Now we introduce a *moduli stack* associated with $Z^{(s)s}$ and describe its relation with $M^{(s)s}$. Let us define the stack

$$\mathfrak{M}^{(s)s} = \mathfrak{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta) := [Z^{(s)s}/SL(V)].$$

Note that $\mathfrak{M}^{(s)s}$ is an algebraic stack of finite type and \mathfrak{M}^s is an open substack of \mathfrak{M}^{ss} .

We explain the relation between $\mathfrak{M}^{(s)s}$ and $M^{(s)s}$. First we recall the notion of *good moduli space* for algebraic stacks.

DEFINITION 3.21. [7, Definition 3.1] A morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if it is quasi-compact and the functor $f_*: \mathrm{QCoh}(\mathcal{X}) \rightarrow \mathrm{QCoh}(\mathcal{Y})$ is exact. ■

DEFINITION 3.22. [7, Definition 4.1 and 7.1] Let $f: \mathcal{X} \rightarrow Y$ be a morphism where \mathcal{X} is an algebraic stack and Y an algebraic space. We say that f is a *good moduli space* if the following properties are satisfied:

- f is cohomologically affine,
- the natural morphism $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_{\mathcal{X}})$ is an isomorphism.

Moreover, a good moduli space f is a *tame moduli space* if the map $[\mathcal{X}(\mathrm{Spec}(k))] \rightarrow Y(\mathrm{Spec}(k))$ is a bijection of sets, where $[\mathcal{X}(\mathrm{Spec}(k))]$ denotes the set of isomorphism classes of objects of $\mathcal{X}(\mathrm{Spec}(k))$. ■

Since the ample line bundle $\mathcal{O}_{Z'}(n_1, n_2)|_{Z^{ss}}$ is $\mathrm{SL}(V)$ -equivariant, it descends to a line bundle $\mathcal{O}(n_1, n_2)$ on \mathfrak{SM}^{ss} . The morphism $\tilde{\pi}$ induces a morphism $\pi_{\mathfrak{S}}: \mathfrak{SM}^{ss} \rightarrow M^{ss}$. By [7, Theorem 13.6] (which is a stacky version of [58, Theorem 4.2.10]), we get the following result.

THEOREM 3.23. [23, Theorem 4.12] *The morphism $\pi_{\mathfrak{S}}: \mathfrak{SM}^{ss} \rightarrow M^{ss}$ is a good moduli space and $\pi_{\mathfrak{S}}^*(\mathcal{O}_{M^{ss}}(1)) \simeq \mathcal{O}(n_1, n_2)^{\otimes l}$. Moreover, the morphism $\pi_{\mathfrak{S}}: \mathfrak{SM}^s \rightarrow M^s$ is a tame moduli space.*

Furthermore, by [7, Theorem 6.6] we can state the following universal property for $\pi_{\mathfrak{S}}: \mathfrak{SM}^{ss} \rightarrow M^{ss}$.

PROPOSITION 3.24. [23, Proposition 4.13] *Let T be an algebraic space and $f: \mathfrak{SM}^{ss} \rightarrow T$ a morphism. There exists a unique morphism $g: M^{ss} \rightarrow T$ such that $f = g \circ \pi_{\mathfrak{S}}$.*

We introduce two more algebraic stacks of finite type

$$\begin{aligned} \mathfrak{M}^{(s)s} &= \mathfrak{M}_{\mathcal{X}/k}^{(s)s}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta) := [Z^{(s)s}/\mathrm{GL}(V)] , \\ \mathfrak{PM}^{(s)s} &= \mathfrak{PM}_{\mathcal{X}/k}^{(s)s}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta) := [Z^{(s)s}/\mathrm{PGL}(V)] . \end{aligned}$$

Note that the stack $\mathfrak{PM}^{(s)s}$ is well defined as the multiplicative group \mathbb{G}_m is contained in the stabilizer of every point of Z^{ss} (cf. Lemma 3.17).

A natural question is if there is a relation between the stacks $\mathfrak{SM}^{(s)s}$, $\mathfrak{M}^{(s)s}$ and $\mathfrak{PM}^{(s)s}$. First, note that the étale groupoid of the étale presentation $Z^{(s)s} \rightarrow \mathfrak{M}^{(s)s}$ is

$$Z^{(s)s} \times \mathrm{GL}(V) \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{p_{Z^{(s)s}}} \end{array} Z^{(s)s} ,$$

where a is the action morphism of $\mathrm{GL}(V)$ on $Z^{(s)s}$. Since \mathbb{G}_m acts on $Z^{(s)s} \times \mathrm{GL}(V)$ by leaving a and $p_{Z^{(s)s}}$ invariant, we can *rigidify* the étale groupoid (the notion of *rigidification* is explained in [1, Section 5])¹ to get

$$Z^{(s)s} \times \mathrm{PGL}(V) \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{p_{Z^{(s)s}}} \end{array} Z^{(s)s} .$$

This is the étale groupoid of $\mathfrak{PM}^{(s)s}$. In particular, $\mathfrak{M}^{(s)s} \rightarrow \mathfrak{PM}^{(s)s}$ is a \mathbb{G}_m -gerbe. On the other hand, we can perform the same procedure on the étale groupoid of $\mathfrak{SM}^{(s)s}$ with respect to the group $\mu(V) \subset \mathrm{SL}(V)$, where $\mu(V)$ is the group of $\dim(V)$ -roots of unity, and we get that the rigidification is isomorphic to $\mathfrak{PM}^{(s)s}$. Hence $\mathfrak{SM}^{(s)s} \rightarrow \mathfrak{PM}^{(s)s}$ is a $\mu(V)$ -gerbe.

The morphism $\pi_{\mathfrak{S}}: \mathfrak{SM}^{(s)s} \rightarrow M^{(s)s}$ induces a morphism $\pi_{\mathfrak{P}}: \mathfrak{PM}^{(s)s} \rightarrow M^{(s)s}$ (cf. [1, Theorem 5.1.5-(2)]), so that we get a morphism $\pi: \mathfrak{M}^{(s)s} \rightarrow M^{(s)s}$ and the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{SM}^{(s)s} & \longrightarrow & \mathfrak{PM}^{(s)s} & \longleftarrow & \mathfrak{M}^{(s)s} \\ & \searrow \pi_{\mathfrak{S}} & \downarrow \pi_{\mathfrak{P}} & \swarrow \pi & \\ & & M^{(s)s} & & \end{array} .$$

¹Note that here we need the most general definition of rigidification, and not just the notion of *rigidification with respect to the generic stabilizer*, as introduced in Section 1.5.3.

Statements as those in Theorem 3.23 hold also for π and $\pi_{\mathfrak{P}}$, cf. [106]. Moreover, according to the proof of [91, Theorem 6.22-(1)], the universal property stated in Proposition 3.24 also holds for π and $\pi_{\mathfrak{P}}$.

Let us denote by $[\mathfrak{M}^{(s)s}]$ the contravariant functor which associates with any scheme S the set $[\mathfrak{M}^{(s)s}](S)$ of isomorphism classes of objects of $\mathfrak{M}^{(s)s}(S)$. The morphism π factors through $\mathfrak{M}^{(s)s} \rightarrow [\mathfrak{M}^{(s)s}]$. To conclude this section we show that the contravariant functors $[\mathfrak{M}^{(s)s}]$ is isomorphic to the *moduli* functor $\underline{\mathcal{M}}^{(s)s}$ of δ -(semi)stable framed sheaves on \mathcal{X} , i.e., the contravariant functor

$$\underline{\mathcal{M}}^{(s)s} = \underline{\mathcal{M}}_{\mathcal{X}/k}^{(s)s}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta): (\text{Sch}/k)^\circ \rightarrow (\text{Sets})$$

which associates with any scheme S of finite type the set of isomorphism classes of flat families of δ -(semi)stable framed sheaves on \mathcal{X} with Hilbert polynomial P_0 parameterized by S .

THEOREM 3.25. [23, Theorem 4.14] *The functor $\underline{\mathcal{M}}^{(s)s}$ is isomorphic to $[\mathfrak{M}^{(s)s}]$.*

We have obtained the following factorization of the structure morphism π :

$$(19) \quad \begin{array}{ccccc} & & \pi & & \\ & \nearrow & & \searrow & \\ \mathfrak{M}^{(s)s} & \xrightarrow{\quad} & [\mathfrak{M}^{(s)s}] & \xrightarrow{\quad} & \mathbf{M}^{(s)s} \\ & & \downarrow \eta & & \uparrow \Psi^{(s)s} \\ & & \underline{\mathcal{M}}^{(s)s} & & \end{array}$$

3.2.3. The moduli spaces of δ -(semi)stable framed sheaves. In this section we prove that $\mathbf{M}^{(s)s}$ is a *moduli space* for the functor $\underline{\mathcal{M}}^{(s)s}$, i.e., $\mathbf{M}^{(s)s}$ corepresents $\underline{\mathcal{M}}^{(s)s}$ (cf. [58], Definition 2.2.1). In addition, thanks to the next Proposition, we can prove that \mathbf{M}^s is a *fine* moduli space for $\underline{\mathcal{M}}^s$, i.e., \mathbf{M}^s represents $\underline{\mathcal{M}}^s$.

PROPOSITION 3.26. [23, Proposition 4.16] *Let $\mathfrak{U}^s = (\mathcal{U}^s, L_{\mathcal{U}^s}, \phi_{\mathcal{U}^s})$ be the universal family of stable framed sheaves on \mathcal{X} parameterized by Z^s . Then \mathcal{U}^s and $L_{\mathcal{U}^s}$ are invariant with respect to the action of the center \mathbb{G}_m of $\text{GL}(V)$.*

Now we are ready to prove the main result of this section.

THEOREM 3.27. [23, Theorem 4.17] *Let \mathcal{X} be a d -dimensional normal projective irreducible stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ and $(\mathcal{G}, \mathcal{O}_X(1))$ a polarization on it. For any framing sheaf \mathcal{F} , stability polynomial δ and numerical polynomial P_0 of degree d , the projective scheme $\mathbf{M}^{ss} = \mathbf{M}_{\mathcal{X}/k}^{ss}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta)$ is a moduli space for the moduli functor*

$$\underline{\mathcal{M}}^{ss} = \underline{\mathcal{M}}_{\mathcal{X}/k}^{ss}(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta).$$

Moreover the quasi-projective scheme $\mathbf{M}^s = \mathbf{M}_{\mathcal{X}/k}^s(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta)$ is a fine moduli space for the moduli functor

$$\mathcal{M}^s = \mathcal{M}_{\mathcal{X}/k}^s(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta).$$

PROOF. Let $\Psi^{(s)s}: \underline{\mathcal{M}}^{(s)s} \rightarrow \mathbf{M}^{(s)s}$ be the natural transformation defined in (19). Let N be a scheme and $\psi: \underline{\mathcal{M}}^{ss} \rightarrow N$ a natural transformation. Then the universal family \mathfrak{U}^{ss} of δ -semistable framed sheaves on \mathcal{X} parameterized by Z^{ss} defines a morphism $f: Z^{ss} \rightarrow N$ which is $\text{SL}(V)$ -invariant due to the $\text{SL}(V)$ -equivariance of \mathfrak{U}^{ss} . Since \mathbf{M}^{ss} is a categorical quotient,

the morphism f factors through a morphism $M^{ss} \rightarrow N$, therefore the natural transformation ψ factors through Ψ^{ss} .

By Lemma 3.17, the stabilizer in $\mathrm{PGL}(V)$ of a closed point in Z^s is trivial. Hence, by Proposition 3.20 and Luna's étale slice Theorem ([58, Theorem 4.2.12]), $Z^s \rightarrow M^s$ is a $\mathrm{PGL}(V)$ -torsor. Since the universal family \mathfrak{U}^s of δ -stable framed sheaves on \mathcal{X} parameterized by Z^s is $\mathrm{PGL}(V)$ -linearized by Proposition 3.26, it descends to a universal family of δ -stable framed sheaves on \mathcal{X} parameterized by M^s . \square

COROLLARY 3.28. [23, corollary 4.16] *The algebraic stack \mathfrak{M}^s is a \mathbb{G}_m -gerbe over its coarse moduli scheme M^s .*

We conclude this section by stating a theorem about the tangent space and the obstruction to the smoothness of the moduli spaces of δ -stable framed sheaves. The proof is just a straightforward generalization of the same result for δ -framed sheaves on smooth projective stacks (cf. [56, Theorem 4.1]), thanks to the result of Olsson and Starr about the tangent space of the Quot scheme for Deligne-Mumford stacks (cf. [93, Lemma 2.5]).

THEOREM 3.29. [23, Theorem 4.19] *Let $[(\mathcal{E}, \phi_{\mathcal{E}})]$ be a point in the moduli space $M_{\mathcal{X}/k}^s(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta)$ of δ -stable framed sheaves on \mathcal{X} . Consider \mathcal{E} and $\phi_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{F}$ as complexes concentrated in degree zero, and zero and one, respectively.*

- (i) *The Zariski tangent space of $M_{\mathcal{X}/k}^s(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta)$ at a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ is naturally isomorphic to the first hyperext group $\mathbb{E}\mathrm{xt}^1(\mathcal{E}, \mathcal{E} \xrightarrow{\phi_{\mathcal{E}}} \mathcal{F})$.*
- (ii) *If the second hyperext group $\mathbb{E}\mathrm{xt}^2(\mathcal{E}, \mathcal{E} \xrightarrow{\phi_{\mathcal{E}}} \mathcal{F})$ vanishes, $M_{\mathcal{X}/k}^s(\mathcal{G}, \mathcal{O}_X(1); P_0, \mathcal{F}, \delta)$ is smooth at $[(\mathcal{E}, \phi_{\mathcal{E}})]$.*

3.2.4. Framed sheaves on projective orbifolds. In this section we assume that \mathcal{X} is a projective orbifold of dimension d . Let $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ be a d -dimensional framed sheaf on \mathcal{X} . The *rank* (resp. the *degree*) of \mathfrak{E} is the rank (resp. the degree) of \mathcal{E} . The *framed degree* of a d -dimensional framed sheaf \mathfrak{E} is

$$\deg_{\mathcal{G}}(\mathfrak{E}) := \deg_{\mathcal{G}}(\mathcal{E}) - \epsilon(\phi_{\mathcal{E}})\delta_1,$$

while its *framed slope* is

$$\mu_{\mathcal{G}}(\mathfrak{E}) := \frac{\deg_{\mathcal{G}}(\mathfrak{E})}{\mathrm{rk}(\mathcal{E})}.$$

Let \mathcal{E}' be a subsheaf of \mathcal{E} with quotient $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$. If \mathcal{E} , \mathcal{E}' and \mathcal{E}'' are d -dimensional, the framed degree of \mathfrak{E} behaves additively, i.e., $\deg_{\mathcal{G}}(\mathfrak{E}) = \deg_{\mathcal{G}}(\mathfrak{E}') + \deg_{\mathcal{G}}(\mathfrak{E}'')$.

DEFINITION 3.30. A d -dimensional framed sheaf $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ is μ -(semi)stable with respect to δ_1 if and only if $\ker \phi_{\mathcal{E}}$ is torsion-free and the following conditions are satisfied:

- (i) $\deg_{\mathcal{G}}(\mathcal{E}') \leq \mathrm{rk}(\mathcal{E}')\mu_{\mathcal{G}}(\mathfrak{E})$ for all subsheaves $\mathcal{E}' \subseteq \ker \phi_{\mathcal{E}}$,
- (ii) $(\deg_{\mathcal{G}}(\mathcal{E}') - \delta_1) \leq \mathrm{rk}(\mathcal{E}')\mu_{\mathcal{G}}(\mathfrak{E})$ for all subsheaves $\mathcal{E}' \subset \mathcal{E}$ with $\mathrm{rk}(\mathcal{E}') < \mathrm{rk}(\mathcal{E})$.

■

DEFINITION 3.31. Let $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ be a framed sheaf of rank zero. If $\phi_{\mathcal{E}}$ is injective, we say that \mathfrak{E} is μ -semistable (indeed, in this case the μ -semistability of the corresponding framed sheaf does not depend on δ_1). Moreover, if the degree of \mathcal{E} is δ_1 , we say that \mathfrak{E} is μ -stable with respect to δ_1 . ■

One has the following relations between (semi)stability properties of framed sheaves on \mathcal{X} :

$$\mu\text{-stability} \Rightarrow \delta\text{-stability} \Rightarrow \delta\text{-semistability} \Rightarrow \mu\text{-semistability}.$$

Thus we can apply the results of the previous sections. In particular we get the following result.

THEOREM 3.32. *Let \mathcal{X} be a d -dimensional projective irreducible orbifold with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ and $(\mathcal{G}, \mathcal{O}_X(1))$ a polarization on it. For any framing sheaf \mathcal{F} , stability polynomial δ and numerical polynomial P_0 of degree d , there exists a fine moduli space parameterizing isomorphism classes of μ -stable framed sheaves on \mathcal{X} with Hilbert polynomial P_0 , which is a quasi-projective scheme.*

3.3. $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on two-dimensional projective orbifolds

In this section we introduce the theory of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on two-dimensional smooth projective irreducible stacks. Our main reference is [23, Section 5]. For the corresponding theory in the case of smooth projective irreducible surfaces see also [20].

Let \mathcal{X} be a two-dimensional smooth projective irreducible stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ a normal projective surface. By [107, Proposition 2.8] X only has finite quotient (hence rational, cf. [67]) singularities.

Fix a one-dimensional integral closed substack $\mathcal{D} \subset \mathcal{X}$ and a locally free sheaf $\mathcal{F}_{\mathcal{D}}$ on it. We call \mathcal{D} a *framing divisor* and $\mathcal{F}_{\mathcal{D}}$ a *framing sheaf*.

DEFINITION 3.33. A $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaf on \mathcal{X} is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where \mathcal{E} is a torsion-free sheaf on \mathcal{X} , locally free in a neighborhood of \mathcal{D} , and $\phi_{\mathcal{E}}: \mathcal{E}|_{\mathcal{D}} \xrightarrow{\sim} \mathcal{F}_{\mathcal{D}}$ is an isomorphism. We call $\phi_{\mathcal{E}}$ a *framing* on \mathcal{E} . ■

A morphism between $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on \mathcal{X} is a morphism between framed sheaves as stated in Definition 3.2.

The assumption of locally freeness of the underlying coherent sheaf \mathcal{E} of a $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ in a neighborhood of \mathcal{D} allows one to prove the next Lemma, which will be useful later on.

LEMMA 3.34. [23, Lemma 5.2] *Let \mathcal{E} be a torsion-free sheaf on \mathcal{X} which is locally free in a neighborhood of \mathcal{D} . If \mathcal{E}' is a saturated coherent subsheaf of \mathcal{E} , the restriction $\mathcal{E}'|_{\mathcal{D}}$ is a subsheaf of $\mathcal{E}|_{\mathcal{D}}$.*

3.3.1. Boundedness. The first result we prove concerns the boundedness of the family of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on \mathcal{X} with fixed Hilbert polynomial. In order to prove it, we need to impose some conditions.

The structure morphism $\pi: \mathcal{X} \rightarrow X$ induces a one-to-one correspondence between integral closed substacks of \mathcal{X} and integral closed subschemes of X in the following way [72]: for any integral closed substack \mathcal{V} of \mathcal{X} , $\pi(\mathcal{V})$ is a closed integral subscheme of X , and *vice versa*, for any integral closed subscheme $V \subset X$, the fibered product $(V \times_X \mathcal{X})_{\text{red}}$ is an integral closed substack of \mathcal{X} . Moreover, V is the coarse moduli scheme of \mathcal{V} (cf. [4, Lemma 2.3.3]).

Let $D := \pi(\mathcal{D})$ be the coarse moduli scheme of \mathcal{D} . In the following we assume that D is a smooth curve. Furthermore, we fix a polarization $(\mathcal{G}, \mathcal{O}_X(1))$ on \mathcal{X} such that \mathcal{G} is a direct sum of powers of a π -ample locally free sheaf.

Note that the maximum $N_{\mathcal{D}}$ of the numbers of the conjugacy classes of any geometric stabilizer group of \mathcal{D} is less or equal to the corresponding number $N_{\mathcal{X}}$ for \mathcal{X} , so that $\mathcal{G}_{|\mathcal{D}}$ is a generating sheaf for \mathcal{D} (cf. Remark 1.8). Thus, also using part (ii) of Proposition 1.12, we obtain that \mathcal{D} is a projective stack.

Our strategy consists in proving that the family $\mathcal{C}_{\mathcal{X}}$ of torsion-free sheaves on \mathcal{X} whose restriction to \mathcal{D} is isomorphic to a fixed locally free sheaf is contained in the family \mathcal{C}_X of torsion-free sheaves on X whose restriction to D is isomorphic to a fixed locally free sheaf. Then the boundedness of the family \mathcal{C}_X ensures the boundedness of the family $\mathcal{C}_{\mathcal{X}}$.

LEMMA 3.35. [23, Lemma 5.3] *Let \mathcal{X} be a two-dimensional smooth projective irreducible stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ a normal projective surface and $(\mathcal{G}, \mathcal{O}_X(1))$ a polarization on it, where \mathcal{G} is a direct sum of powers of a π -ample locally free sheaf. Fix a one-dimensional integral closed substack $\mathcal{D} \subset \mathcal{X}$, whose coarse moduli space $\mathcal{D} \rightarrow D$ is a smooth curve, and a locally free sheaf $\mathcal{F}_{\mathcal{D}}$ on it. Let \mathcal{E} be a torsion-free sheaf on \mathcal{X} such that $\mathcal{E}_{|\mathcal{D}} \simeq \mathcal{F}_{\mathcal{D}}$. Then $F_{\mathcal{G}}(\mathcal{E})$ is a torsion-free sheaf on X and $F_{\mathcal{G}}(\mathcal{E})|_D \simeq F_{\mathcal{G}_{|\mathcal{D}}}(\mathcal{F}_{\mathcal{D}})$ is an isomorphism, where $F_{\mathcal{G}_{|\mathcal{D}}}(\mathcal{F}_{\mathcal{D}})$ is a locally free sheaf on D .*

PROOF. Let us consider the short exact sequence

$$0 \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \rightarrow \mathcal{E} \rightarrow i_*(\mathcal{F}_{\mathcal{D}}) \rightarrow 0 .$$

Since the functor $F_{\mathcal{G}}$ is exact, we get

$$0 \rightarrow F_{\mathcal{G}}(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})) \rightarrow F_{\mathcal{G}}(\mathcal{E}) \rightarrow \iota_*(F_{\mathcal{G}_{|\mathcal{D}}}(\mathcal{F}_{\mathcal{D}})) \rightarrow 0 ,$$

where $i: \mathcal{D} \hookrightarrow \mathcal{X}$ and $\iota: D \hookrightarrow X$ are the inclusion morphisms.

By Proposition 1.22, $F_{\mathcal{G}}(\mathcal{E})$ (resp. $F_{\mathcal{G}_{|\mathcal{D}}}(\mathcal{F}_{\mathcal{D}})$) is a torsion-free sheaf on X (resp. D). Since D is a smooth irreducible projective curve, $F_{\mathcal{G}_{|\mathcal{D}}}(\mathcal{F}_{\mathcal{D}})$ is locally free. Now $\text{Supp}(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}))$ is disjoint from \mathcal{D} , so that, by Corollary 1.23 the support of $F_{\mathcal{G}}(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}))$ is disjoint from D as well. Then $F_{\mathcal{G}}(\mathcal{E})|_D \simeq F_{\mathcal{G}_{|\mathcal{D}}}(\mathcal{F}_{\mathcal{D}})$. \square

DEFINITION 3.36. An effective irreducible \mathbb{Q} -Cartier divisor D in X is a *good framing divisor* if there exists $a_D \in \mathbb{N}_{>0}$ such that $a_D D$ is a big and nef Cartier divisor on X (i.e., $a_D D$ is a nef Cartier divisor, and $(a_D D)^2 > 0$). \blacksquare

THEOREM 3.37. [23, Theorem 5.5] *Let X be a normal irreducible projective surface with rational singularities and H an effective ample divisor on it. Let D be a good framing divisor in X which contains the singular locus of X , and F_D a locally free sheaf on D . Then for any numerical polynomial $P \in \mathbb{Q}[n]$ of degree two, the family \mathcal{C} of torsion-free sheaves E on X such that $E|_D \simeq F_D$ and $P(E) = P$ is bounded.*

PROOF. We shall adapt the arguments of [74], Theorem 3.2.4. We want to apply Kleiman's criterion ([58, Theorem 1.7.8]), so that we need to determine upper bounds for the quantities $h^0(X, E)$ and $h^0(H, E|_H)$, for any torsion-free sheaf E in the family \mathcal{C} .

Let us fix a torsion-free sheaf E on X such that $E|_D \simeq F_D$ and $P(E) = P$. Consider the short exact sequence

$$0 \rightarrow E(-(\nu+1)D) \rightarrow E(-\nu D) \rightarrow (E(-\nu D))|_D \rightarrow 0 .$$

By induction, we get $h^0(X, E) \leq h^0(X, E(-nD)) + \sum_{\nu=0}^{n-1} h^0(D, F_D(-\nu D))$ for all $n \geq 1$. Let $n = ma_D + t$ with $0 \leq t \leq a_D - 1$, then by [73, Theorem 1.4.37], we have $h^0(X, E(-nD)) = O(m^2)$. Since $\mathcal{O}_X(a_D D)$ is big and nef, the line bundle $\mathcal{O}_X(a_D D)|_D$ is ample, hence there exists a positive integer ν_0 such that for any $\nu \geq \nu_0$ one has $h^0(D, F_D(-\nu D)|_D) = 0$. Set $K = \sum_{\nu=0}^{\nu_0-1} h^0(D, F_D(-\nu D))$. This is independent of E and

$$(20) \quad h^0(X, E) \leq K .$$

We want to estimate $h^0(H, E|_H)$. Since $h^0(H, E|_H) \leq h^0(X, E) + h^1(X, E(-H))$, we need only to estimate the quantities on the right-hand-side. First, note that $h^1(X, E(-H)) = h^0(X, E(-H)) + h^2(X, E(-H)) - \chi(X, E(-H))$. Since the Hilbert polynomial of E is fixed, $\chi(X, E(-H)) = P(-1)$. Moreover, the restriction of $E(-H)$ to D is the fixed locally free sheaf $F_D \otimes \mathcal{O}_X(-H)|_D$, so we can adapt the previous arguments to obtain

$$h^0(X, E(-H)) \leq L ,$$

for some positive integer L . Now we just need an estimate of $h^2(X, E(-H))$. Set $G = E(-H)$. By Serre duality ([54, Theorem III-7.6]), $H^2(X, G) \simeq \text{Hom}(G, \omega_X)^\vee$, where ω_X is the dualizing sheaf of X . Let $\pi: \hat{X} \rightarrow X$ be a resolution of singularities of X . Then we have the map

$$\text{Hom}(G, \omega_X) \xrightarrow{\pi^*} \text{Hom}(\pi^* G, \pi^* \omega_X) .$$

This map is injective. Indeed let $\varphi: G \rightarrow \omega_X$ be a morphism such that $\pi^* \varphi = 0$. Since π is an isomorphism over X_{sm} , the sheaf $\text{Im}(\varphi)$ is supported on the singular locus $\text{sing}(X)$. Since ω_X is a torsion free sheaf of rank one (cf. e.g. [99, Appendix 1]), $\varphi = 0$.

By Kempf's Theorem ([66, Chapter I.3]) we have $\pi_* \omega_{\hat{X}} \simeq \omega_X$, hence a morphism $\pi^* \omega_X \rightarrow \omega_{\hat{X}}$, and maps

$$\text{Hom}(G, \omega_X) \rightarrow \text{Hom}(\pi^* G, \pi^* \omega_X) \rightarrow \text{Hom}(\pi^* G, \omega_{\hat{X}}) .$$

The kernel of the composition $f: \text{Hom}(G, \omega_X) \rightarrow \text{Hom}(\pi^* G, \omega_{\hat{X}})$ lies in $\text{Hom}(\pi^* G, T)$, where T is the torsion of $\pi^* \omega_X$. Since the singularities of X are in D , the group $\text{Hom}(\pi^* G, T)$ injects into $\text{Hom}(\pi^* G|_{\hat{D}}, T|_{\hat{D}})$, where $\hat{D} = \pi^{-1}(D)$. The dimension M of the latter group does not depend on E since $(\pi^* G)|_{\hat{D}} \simeq \pi^*(F_D \otimes \mathcal{O}_X(-H)|_D)$. Thus $\dim \ker f \leq \dim \text{Hom}(\pi^* G, T) \leq M$.

Note that $\pi^* G$ is torsion-free since G is locally free in a neighborhood of D , and D contains the singular locus of X . Consider the exact sequence

$$0 \rightarrow \pi^* G \rightarrow (\pi^* G)^{\vee\vee} \rightarrow Q \rightarrow 0 ,$$

where the support of Q is zero-dimensional. By applying the functor $\text{Hom}(\cdot, \omega_{\hat{X}})$ one gets $\text{Hom}(\pi^* G, \omega_{\hat{X}}) \simeq \text{Hom}((\pi^* G)^{\vee\vee}, \omega_{\hat{X}})$. The dual $(\pi^* G)^\vee$ is locally free, so that

$$\dim \text{Hom}((\pi^* G)^{\vee\vee}, \omega_{\hat{X}}) = \dim \text{Hom}(\mathcal{O}_{\hat{X}}, (\pi^* G)^\vee \otimes \omega_{\hat{X}}) = h^0((\pi^* G)^\vee \otimes \omega_{\hat{X}}) .$$

Moreover, \hat{D} is a good framing divisor, since it is a pullback by a birational morphism, and $(\pi^*(G)^\vee \otimes \omega_{\hat{X}})|_{\hat{D}}$ is a fixed locally free sheaf on \hat{D} , so that we can use the same argument as before, and obtain

$$\dim \operatorname{Hom}((\pi^*G)^{\vee\vee}, \omega_{\hat{X}}) \leq N$$

for some constant $N > 0$. Then the dimension of the image of f is bounded by N . Therefore, $h^2(X, E(-H)) = \dim \operatorname{Hom}(E(-H), \omega_X) \leq M + N$. Thus

$$h^0(H, E|_H) \leq h^0(X, E) + h^1(X, E(-H)) \leq K + L + M + N - P(-1) =: K'.$$

Thus by Kleiman's criterion, the family \mathcal{C} is bounded. \square

THEOREM 3.38. [23, Theorem 5.6] *Let \mathcal{X} be a two-dimensional smooth projective irreducible stack with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ a normal projective surface and $(\mathcal{G}, \mathcal{O}_X(1))$ a polarization on it, where \mathcal{G} is a direct sum of powers of a π -ample locally free sheaf. Fix a one-dimensional integral closed substack $\mathcal{D} \subset \mathcal{X}$ and a locally free sheaf $\mathcal{F}_{\mathcal{D}}$ on it. Assume that the coarse moduli space $\mathcal{D} \rightarrow D$ of \mathcal{D} is a smooth curve containing the singular locus of X and is a good framing divisor. For any numerical polynomial $P \in \mathbb{Q}[n]$ of degree two, the family $\mathcal{C}_{\mathcal{X}}$ of torsion-free sheaves \mathcal{E} on \mathcal{X} such that $\mathcal{E}|_{\mathcal{D}} \simeq \mathcal{F}_{\mathcal{D}}$ and $P_{\mathcal{G}}(\mathcal{E}) = P$ is bounded.*

PROOF. By using [91, Corollary 4.17], one has that $\mathcal{C}_{\mathcal{X}}$ is bounded if and only if the family \mathcal{C}_X of torsion-free sheaves $F_{\mathcal{G}}(\mathcal{E})$ on X such that $F_{\mathcal{G}}(\mathcal{E})|_D \simeq F_{\mathcal{G}}(\mathcal{F}_{\mathcal{D}})$ and $P(F_{\mathcal{G}}(\mathcal{E})) = P$ is bounded. This is a subfamily of the family of torsion-free sheaves E on X with Hilbert polynomial P such that $E|_D \simeq F_{\mathcal{G}}(\mathcal{F}_{\mathcal{D}})$. This latter family is bounded by Theorem 3.37. \square

3.3.2. Stability of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves. In this section we shall show that any $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaf on \mathcal{X} is μ -stable with respect to a suitable choice of an effective ample divisor on X and of the parameter δ_1 . From now on, we assume that \mathcal{X} is an orbifold and \mathcal{D} is smooth. As in the previous section, we assume that the coarse moduli space $\mathcal{D} \rightarrow D$ of \mathcal{D} is a smooth curve and a good framing divisor.

Since $D \cap X_{sm}$ is an irreducible effective Cartier divisor, where X_{sm} is the smooth locus of X , there exists a unique positive integer $a_{\mathcal{D}}$ such that $\pi^{-1}(D \cap X_{sm}) = a_{\mathcal{D}}(\mathcal{D} \cap \pi^{-1}(X_{sm}))$. Then $\pi^{-1}(a_D D \cap X_{sm}) = a_{\mathcal{D}} a_D (\mathcal{D} \cap \pi^{-1}(X_{sm}))$. Since X is normal, $\operatorname{codim}(X \setminus X_{sm}) \geq 2$; moreover, since π is a *codimension preserving morphism* (cf. [39, Remark 4.3]), also $\operatorname{codim}(\pi^{-1}(X \setminus X_{sm}))$ is at least two and therefore

$$(21) \quad \pi^* \mathcal{O}_X(a_D D) \simeq \mathcal{O}_{\mathcal{X}}(\mathcal{D})^{a_{\mathcal{D}} a_D}.$$

This isomorphism will be useful later on.

DEFINITION 3.39. Let \mathcal{X} be an orbifold and \mathcal{D} a smooth integral closed substack of \mathcal{X} such that $D := \pi(\mathcal{D})$ is a good framing divisor. A *good framing sheaf* on \mathcal{D} is a locally free sheaf $\mathcal{F}_{\mathcal{D}}$ for which there exists a real positive number A_0 , with

$$0 \leq A_0 < \frac{1}{\operatorname{rk}(\mathcal{F}_{\mathcal{D}})} \int_{\mathcal{X}}^{et} c_1^{et}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})) \cdot c_1^{et}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})) = \frac{1}{\operatorname{rk}(\mathcal{F}_{\mathcal{D}})} \cdot \frac{(a_D D)^2}{a_D^2 k_{\mathcal{D}}^2},$$

where $k_{\mathcal{D}}$ is the order of the generic stabilizer of \mathcal{D} , such that for any locally free subsheaf \mathcal{F}' of $\mathcal{F}_{\mathcal{D}}$ we have

$$\frac{1}{\operatorname{rk}(\mathcal{F}')} \int_{\mathcal{D}}^{et} c_1^{et}(\mathcal{F}') \leq \frac{1}{\operatorname{rk}(\mathcal{F}_{\mathcal{D}})} \int_{\mathcal{D}}^{et} c_1^{et}(\mathcal{F}_{\mathcal{D}}) + A_0.$$

■

REMARK 3.40. Note that if $\mathcal{F}_{\mathcal{D}}$ is a line bundle on \mathcal{D} , trivially it is a good framing sheaf. Moreover a direct sum of line bundles \mathcal{L}_i such that the value of

$$\int_{\mathcal{D}}^{et} c_1^{et}(\mathcal{L}_i)$$

is the same for all i is a good framing sheaf. \triangle

Let H be an ample divisor on X ; then $H_n = H + na_D D$ is ample for any positive integer n . Let \mathcal{G} a generating sheaf on \mathcal{X} . In the following we would like to compare the degree of a coherent sheaf \mathcal{E} on \mathcal{X} with respect to the polarizations $(\mathcal{G}, \mathcal{O}_X(H))$ and $(\mathcal{G}, \mathcal{O}_X(H_n))$. To avoid confusion, we shall write explicitly what polarization we use to compute the coefficients of the Hilbert polynomial.

Since \mathcal{X} is smooth, to compute the degree of a coherent sheaf on \mathcal{X} we can use the Töen-Riemann-Roch theorem (cf. Appendix B). In the following we shall use the notation of the Appendix: when we shall consider cohomology classes of $H_{rep}^\bullet(\mathcal{X}) := H_{et}^\bullet(\mathcal{I}(\mathcal{X}))$, there will be the label “rep”; on the other hand for classes in $H_{et}^\bullet(\mathcal{X})$ we shall use the label “et”.

As explained in Appendix B, there is a decomposition

$$(22) \quad H_{rep}^\bullet(\mathcal{X}) = H_{et}^\bullet(\mathcal{I}(\mathcal{X})) \cong H_{et}^\bullet(\mathcal{X}) \oplus H_{et}^\bullet(\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}) .$$

For any class $\alpha \in H_{rep}^\bullet(\mathcal{X})$ we denote by $\alpha = \alpha_1 + \alpha_{\neq 1}$ the corresponding decomposition.

Now we introduce the following condition on \mathcal{G} :

CONDITION 3.41. The number

$$\int_{\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}}^{et} \left[\left(\text{ch}^{rep}(\mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{X}}^{\oplus \text{rk}(\mathcal{E})}) \right) \text{ch}^{rep}(\mathcal{G}^\vee) c_1^{rep}(\pi^* \mathcal{L}) \text{td}^{rep}(\mathcal{X}) \right]_{\neq 1}$$

is zero for all coherent sheaves \mathcal{E} on \mathcal{X} and all ample line bundles \mathcal{L} on X .

REMARK 3.42. Recall that \mathcal{X} is an orbifold, i.e., $\mathcal{I}(\mathcal{X})$ has exactly one two-dimensional component, which is \mathcal{X} itself. We point out also that if $\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}$ has no one-dimensional components, so the previous condition is trivially satisfied. If $\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}$ has one-dimensional components, the previous condition can be restated by saying that the zero degree part of

$$\left(\text{ch}^{rep}(\mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{X}}^{\oplus \text{rk}(\mathcal{E})}) \right) \text{ch}^{rep}(\mathcal{G}^\vee)$$

is zero over the one-dimensional part of $\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}$ for any coherent sheaf \mathcal{E} on \mathcal{X} . \triangle

LEMMA 3.43. [23, Lemma 5.10] *Let \mathcal{X} be a two-dimensional projective irreducible orbifold with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ a normal projective surface and \mathcal{G} a generating sheaf for it. Assume that condition 3.41 holds. Fix a one-dimensional smooth integral closed substack $\mathcal{D} \subset \mathcal{X}$, whose coarse moduli space $\mathcal{D} \rightarrow D$ is a smooth curve and a good framing divisor. Let H be an ample divisor on X and set $H_n = H + na_D D$ for any positive integer n . Then for any coherent sheaf \mathcal{E} we have*

$$\deg_{\mathcal{G}, H_n}(\mathcal{E}) = \deg_{\mathcal{G}, H}(\mathcal{E}) + na_D a_{\mathcal{D}} \text{rk}(\mathcal{G}) \int_{\mathcal{D}}^{et} c_1^{et}(\mathcal{E}|_{\mathcal{D}}) .$$

PROOF. By the Töen-Riemann-Roch theorem the degree of \mathcal{E} with respect to the polarization $(\mathcal{G}, \mathcal{O}_X(H_n))$ is

$$\deg_{\mathcal{G}, H_n}(\mathcal{E}) = \int_{\mathcal{X}}^{rep} \left(\text{ch}^{rep}(\mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{X}}^{\oplus \text{rk}(\mathcal{E})}) \right) \text{ch}^{rep}(\mathcal{G}^{\vee}) c_1^{rep}(\pi^* \mathcal{O}_X(H_n)) \text{td}^{rep}(\mathcal{X}) .$$

Using the decomposition (22) and Condition 3.41, we obtain

$$\deg_{\mathcal{G}, H_n}(\mathcal{E}) = \int_{\mathcal{X}}^{et} \left[\left(\text{ch}^{rep}(\mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{X}}^{\oplus \text{rk}(\mathcal{E})}) \right) \text{ch}^{rep}(\mathcal{G}^{\vee}) c_1^{rep}(\pi^* \mathcal{O}_X(H_n)) \text{td}^{rep}(\mathcal{X}) \right]_1 .$$

By Formula (141) in Appendix B and the identity $c_1^{et}(\pi^* \mathcal{O}_X(H_n)) = \pi^* c_1^{et}(\mathcal{O}_X(H_n))$, we have

$$\deg_{\mathcal{G}, H_n}(\mathcal{E}) = \int_{\mathcal{X}}^{et} \left(\text{ch}^{et}(\mathcal{E}) - \text{ch}^{et}(\mathcal{O}_{\mathcal{X}}^{\oplus \text{rk}(\mathcal{E})}) \right) \text{ch}^{et}(\mathcal{G}^{\vee}) \pi^* c_1^{et}(\mathcal{O}_X(H_n)) \text{td}^{et}(\mathcal{X}) .$$

Since the zero degree part of $\text{ch}^{et}(\mathcal{E})$ is $\text{rk}(\mathcal{E})$, the degree becomes

$$\deg_{\mathcal{G}, H_n}(\mathcal{E}) = \text{rk}(\mathcal{G}) \int_{\mathcal{X}}^{et} c_1^{et}(\mathcal{E}) \pi^* c_1^{et}(\mathcal{O}_X(H_n)) .$$

Moreover $c_1^{et}(\mathcal{O}_X(H_n)) = c_1^{et}(\mathcal{O}_X(H)) + n c_1^{et}(\mathcal{O}_X(a_D D))$, so that we have

$$\deg_{\mathcal{G}, H_n}(\mathcal{E}) = \deg_{\mathcal{G}, H}(\mathcal{E}) + \text{rk}(\mathcal{G}) \int_{\mathcal{X}}^{et} c_1^{et}(\mathcal{E}) \pi^* c_1^{et}(\mathcal{O}_X(n a_D D)) .$$

By Formula (21) we get

$$\int_{\mathcal{X}}^{et} c_1^{et}(\mathcal{E}) \pi^* c_1^{et}(\mathcal{O}_X(a_D D)) = a_D a_{\mathcal{D}} \int_{\mathcal{X}}^{et} c_1^{et}(\mathcal{E}) c_1^{et}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})) = a_D a_{\mathcal{D}} \int_{\mathcal{D}} c_1^{et}(\mathcal{E}|_{\mathcal{D}}) .$$

Thus we obtain the assertion. \square

By using similar computations we also get the following result.

LEMMA 3.44. [23, Lemma 5.11] *Under the same hypotheses of Lemma 3.43 we have*

$$\deg_{\mathcal{G}, H_n}(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D})) = \deg_{\mathcal{G}, H_n}(\mathcal{E}) + \text{rk}(\mathcal{E}) \deg_{\mathcal{G}, H_n}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})) .$$

THEOREM 3.45. [23, Theorem 5.12] *Let \mathcal{X} be a two-dimensional projective irreducible orbifold with coarse moduli scheme $\pi: \mathcal{X} \rightarrow X$ a normal projective surface and \mathcal{G} a generating sheaf given as direct sum of powers of a π -ample locally free sheaf. Assume that condition 3.41 holds. Fix a one-dimensional smooth integral closed substack $\mathcal{D} \subset \mathcal{X}$, whose coarse moduli space $\mathcal{D} \rightarrow D$ is a smooth curve containing the singular locus of X and a good framing divisor. Let $\mathcal{F}_{\mathcal{D}}$ be a good framing sheaf on \mathcal{D} . Then for any numerical polynomial $P \in \mathbb{Q}[n]$ of degree two, there exist an effective ample divisor C on X and a positive rational number δ_1 such that all the $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on \mathcal{X} with Hilbert polynomial P are μ -stable with respect to δ_1 and the polarization $(\mathcal{G}, \mathcal{O}_X(C))$.*

PROOF. By arguing along the lines of the proof of the analogous theorem for framed sheaves on smooth projective surfaces ([20], Theorem 3.1), and using many of the results so far proved in this section, we get the assertion. Indeed, let H be an effective ample divisor on X and let n be a positive integer. Set $H_n = H + n a_D D$. From now on, we shall use the polarizations (\mathcal{G}, H) and (\mathcal{G}, H_n) on \mathcal{X} .

Let us fix a numerical polynomial P of degree two. The family of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ with Hilbert polynomial P on \mathcal{X} is bounded by Theorem 3.38. Then by the

stacky version of Grothendieck Lemma (cf. [91, Lemma 4.13]) and the Equation (5), there exists a nonnegative constant A_1 , depending only on $\mathcal{F}_{\mathcal{D}}$, P , H , such that for any $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaf $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ with Hilbert polynomial P on \mathcal{X} and for any nonzero subsheaf $\mathcal{E}' \subset \mathcal{E}$

$$\mu_{\mathcal{G}, H}(\mathcal{E}') < \mu_{\mathcal{G}, H}(\mathcal{E}) + A_1 .$$

Now we check that there exists n such that the range of positive rational numbers δ_1 , for which all the $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves with Hilbert polynomial P on \mathcal{X} are μ -stable with respect to δ_1 and the polarization (\mathcal{G}, H_n) , is nonempty.

Let $\mathfrak{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ be a $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaf with Hilbert polynomial P and rank r . Let \mathcal{E}' be a nonzero coherent subsheaf of rank r' of \mathcal{E} . Assume that $\mathcal{E}' \not\subset \ker \phi_{\mathcal{E}}$, so that in addition we can assume that $0 < r' < r$. The μ -stability condition with respect to δ_1 and H_n for \mathcal{E} reads

$$(23) \quad \mu_{\mathcal{G}, H_n}(\mathcal{E}') < \mu_{\mathcal{G}, H_n}(\mathcal{E}) + \left(\frac{1}{r'} - \frac{1}{r} \right) \delta_1 .$$

Since the degree of the saturation of \mathcal{E}' is larger than the degree of \mathcal{E}' , we can replace \mathcal{E}' by its saturation (cf. Remark 1.27). By Lemma 3.34, $\mathcal{E}'_{|\mathcal{D}} \subset \mathcal{E}_{|\mathcal{D}}$. By Lemma 3.43, we obtain

$$\begin{aligned} \mu_{H_n}(\mathcal{E}') &= \frac{1}{r'} \deg_{\mathcal{G}, H_n}(\mathcal{E}') = \mu_{\mathcal{G}, H}(\mathcal{E}') + \frac{n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G})}{r'} \int_{\mathcal{D}} c_1(\mathcal{E}'_{|\mathcal{D}}) \\ &\leq \mu_{H_n}(\mathcal{E}) + n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G}) A_0 + A_1 . \end{aligned}$$

This implies the inequality (23) whenever

$$(24) \quad \frac{r r'}{r - r'} (n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G}) A_0 + A_1) < \delta_1 .$$

If the rank r' of $\mathcal{E}' \subset \ker \phi_{\mathcal{E}} \simeq \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ satisfies $0 < r' < r$, the μ -stability condition for \mathcal{E} is

$$(25) \quad \mu_{\mathcal{G}, H_n}(\mathcal{E}') < \mu_{\mathcal{G}, H_n}(\mathcal{E}) - \frac{1}{r} \delta_1 .$$

As before, we can assume that \mathcal{E}' is a saturated subsheaf of $\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$. Therefore by Lemma 3.34, $\mathcal{E}'_{|\mathcal{D}} \subset \mathcal{E}_{|\mathcal{D}} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})_{|\mathcal{D}}$. By Lemma 3.44, the inclusion $\mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \hookrightarrow \mathcal{E}$ yields

$$\mu_{\mathcal{G}, H_n}(\mathcal{E}') \leq \mu_{\mathcal{G}, H_n}(\mathcal{E}) - \deg_{\mathcal{G}, H_n}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})) + n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G}) A_0 + A_1 .$$

Since

$$\deg_{\mathcal{G}, H_n}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})) = \frac{\operatorname{rk}(\mathcal{G})(a_D D) \cdot H}{a_D k_{\mathcal{D}}} + \frac{n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G})(a_D D)^2}{a_D^2 k_{\mathcal{D}}^2} ,$$

we get

$$\mu_{\mathcal{G}, H_n}(\mathcal{E}') \leq \mu_{\mathcal{G}, H_n}(\mathcal{E}) - n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G}) \left(\frac{(a_D D)^2}{a_D^2 k_{\mathcal{D}}^2} - A_0 \right) + A_1 - \frac{\operatorname{rk}(\mathcal{G})(a_D D) \cdot H}{a_D k_{\mathcal{D}}} .$$

We see that this inequality implies the inequality (25) whenever

$$(26) \quad \delta_1 < r \left[n a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G}) \left(\frac{(a_D D)^2}{a_D^2 k_{\mathcal{D}}^2} - A_0 \right) - A_1 + \frac{\operatorname{rk}(\mathcal{G})(a_D D) \cdot H}{a_D k_{\mathcal{D}}} \right] .$$

Let $\mathcal{E}' \subset \ker \phi_{\mathcal{E}} \simeq \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ of rank r . By saturating \mathcal{E}' inside $\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$, we can take $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$. Hence $\mu_{\mathcal{G}, H_n}(\mathcal{E}') = \mu_{\mathcal{G}, H_n}(\mathcal{E}) - \deg_{\mathcal{G}, H_n}(\mathcal{O}_{\mathcal{X}}(\mathcal{D}))$. In this case, the inequality (25) is satisfied for

$$\delta_1 < r \left[n \frac{a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G})(a_D D)^2}{a_D^2 k_{\mathcal{D}}^2} + \frac{\operatorname{rk}(\mathcal{G})(a_D D) \cdot H}{a_D k_{\mathcal{D}}} \right].$$

Note that the inequality (26) implies this latter inequality. Hence the inequalities (24) and (26), for all $r' = 1, \dots, r-1$, have a nonempty interval of common solutions δ_1 if

$$n > \max \left\{ \frac{r A_1 - \frac{\operatorname{rk}(\mathcal{G})(a_D D) \cdot H}{a_D k_{\mathcal{D}}}}{a_D a_{\mathcal{D}} \operatorname{rk}(\mathcal{G}) \left(\frac{(a_D D)^2}{a_D^2 k_{\mathcal{D}}^2} - r A_0 \right)}, 0 \right\}.$$

□

REMARK 3.46. When $\mathcal{X} = X$ is a smooth projective surface and $\mathcal{G} \simeq \mathcal{O}_X$, this proof reduces to the proof of [20], Theorem 3.1. \triangle

By Theorems 3.29 and 3.45 we eventually have:

COROLLARY 3.47. [23, Corollary 5.14] *Under the same assumptions as in Theorem 3.45, there exists a fine moduli space $M_{\mathcal{X}/k}(P_0, \mathcal{D}, \mathcal{F}_{\mathcal{D}})$ parameterizing isomorphism classes of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves $(\mathcal{E}, \phi_{\mathcal{E}})$ on \mathcal{X} with Hilbert polynomial P , which is a quasi-projective scheme. If $\operatorname{Ext}^2(\mathcal{E}, \mathcal{E}(-\mathcal{D})) = 0$ for all the points $[(\mathcal{E}, \phi_{\mathcal{E}})]$, the moduli space $M_{\mathcal{X}/k}(P_0, \mathcal{D}, \mathcal{F}_{\mathcal{D}})$ is a smooth quasi-projective variety.*

REMARK 3.48. Since the moduli space $M_{\mathcal{X}/k}(P_0, \mathcal{D}, \mathcal{F}_{\mathcal{D}})$ is fine, there exists a *universal* flat family $(\tilde{\mathcal{E}}, L_{\tilde{\mathcal{E}}}, \phi_{\tilde{\mathcal{E}}})$ of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on \mathcal{X} parameterized by $M_{\mathcal{X}/k}(P_0, \mathcal{D}, \mathcal{F}_{\mathcal{D}})$. The fact the framing of a $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaf is an isomorphism after restricting to \mathcal{D} implies that $\tilde{\phi}_{\tilde{\mathcal{E}}}: \tilde{\mathcal{E}} \rightarrow p_{\mathcal{X}}^* \mathcal{F}_{\mathcal{D}}$ is an isomorphism over $M_{\mathcal{X}/k}(P_0, \mathcal{D}, \mathcal{F}_{\mathcal{D}}) \times \mathcal{D}$. Moreover, this allows one to dispose of the homothety in the definition of morphisms of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves, so that the line bundle $L_{\tilde{\mathcal{E}}}$ can be taken trivial. \triangle

3.4. $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on two-dimensional projective root toric orbifolds

In this section we apply the theory of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves developed in the previous section to the case of toric orbifolds, as in [23, Section 6]. Let $\pi^{can}: \mathcal{X}^{can} \rightarrow X$ be the *canonical toric orbifold* of a normal projective toric surface X and $\tilde{\mathcal{D}} \subset \mathcal{X}^{can}$ a smooth divisor whose coarse moduli scheme D is a torus-invariant rational curve in X containing the singular locus of X . By performing a *k-root construction* (cf Section 1.2) on \mathcal{X}^{can} along $\tilde{\mathcal{D}}$ we obtain a two-dimensional projective toric orbifold \mathcal{X} , with coarse moduli scheme X , endowed with a smooth divisor \mathcal{D} which is a μ_k -gerbe over $\tilde{\mathcal{D}}$. We shall show that if $\mathcal{O}_{\mathcal{X}^{can}}(\tilde{\mathcal{D}})$ is π^{can} -ample and D is a good framing divisor, Theorem 3.45 holds for any choice of a good framing sheaf $\mathcal{F}_{\mathcal{D}}$ on \mathcal{D} ; hence for any numerical polynomial P of degree two, there exists a fine moduli space parameterizing isomorphism classes of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on \mathcal{X} with Hilbert polynomial P .

In the following, we set $k = \mathbb{C}$.

Let X be a normal projective toric surface, acted on by a torus $\mathbb{C}^* \times \mathbb{C}^*$, and let Σ be its fan in $N_{\mathbb{Q}}$. Since X is projective, the rays of Σ generate $N_{\mathbb{Q}}$. Let $n+2$ be the number

of rays of Σ for some positive integer n . By the orbit-cone correspondence there exist $n + 2$ torus-invariant rational curves D_0, \dots, D_{n+1} . We shall use also the letter D to denote the curve D_{n+1} .

The singular points of X are necessarily torus-invariant, and, by the normality assumption, the singular locus $\text{sing}(X)$ is zero-dimensional, i.e., $\text{sing}(X)$ consists of a finite number of torus-fixed points. We assume that $\text{sing}(X)$ is contained inside D . Then $\text{sing}(X)$ consists at most of the two torus-fixed points of D , which we shall denote by $0, \infty$. Moreover, the complementary set $X_0 := X \setminus D$ is a smooth quasi-projective toric surface. Let us assume that the intersection point of D_0 and D is 0 and the intersection point of D_n and D is ∞ .

Let $\pi^{can}: \mathcal{X}^{can} \rightarrow X$ be the canonical toric orbifold of X . Since π^{can} is an isomorphism over X_{sm} , the “orbifold” structure of \mathcal{X}^{can} lies (at most) at the *stacky* points $\tilde{p}_0 := (\pi^{can})^{-1}(0)_{red}$ and $\tilde{p}_\infty := (\pi^{can})^{-1}(\infty)_{red}$. So we have that $\tilde{\mathcal{D}}_i \simeq D_i$ for $i = 1, \dots, n-1$ and $\tilde{\mathcal{D}}_j$ is an orbifold for $j = 0, n, n+1$. Since the coarse moduli scheme of $\tilde{\mathcal{D}}_j$ is \mathbb{P}^1 , the stack $\tilde{\mathcal{D}}_j$ is a so-called *spherical orbicurve* (cf. [13, Section 5]) for $j = 0, n, n+1$. Since the number of orbifold points is at most two, by [13], Prop. 5.5, we have that

$$(27) \quad \tilde{\mathcal{D}}_0 \simeq \mathcal{F}(a_0, 1), \quad \tilde{\mathcal{D}}_n \simeq \mathcal{F}(a_\infty, 1), \quad \tilde{\mathcal{D}}_{n+1} \simeq \mathcal{F}(a_0, a_\infty),$$

where we denote by $\mathcal{F}(p, q)$ the *football* with two orbifold points of order p and q respectively, where p and q are positive integers. A football is a one-dimensional complete orbifold with coarse moduli scheme \mathbb{P}^1 and at most two orbifold points. Note that $\mathcal{F}(1, 1) \simeq \mathbb{P}^1$.

A well-known consequence of the construction of the coarse moduli space is the existence for any geometric point p of \mathcal{X}^{can} with image x in X of an étale neighborhood $U \rightarrow X$ of x such that $U \times_X \mathcal{X}^{can}$ is a neighborhood of p and is a quotient stack of the form $[Y/\text{Stab}(p)]$, where Y is a scheme. In particular, there is an étale neighborhood U of 0 in X such that $U \times_X \mathcal{X}^{can}$ is an étale neighborhood of \tilde{p}_0 and is a quotient stack of the form $[V/\mu_{a_0}]$, where V is a smooth variety. Then $U = V/\mu_{a_0}$. So a_0 is the order of the singularity of X at 0 . Similarly, a_∞ is the order of the singularity of X at ∞ .

Since all toric footballs are fibered products of root stacks over \mathbb{P}^1 (cf. [39], Example 7.31), we get that

$$(28) \quad \tilde{\mathcal{D}}_0 \simeq {}^a\sqrt{0/\mathbb{P}^1}, \quad \tilde{\mathcal{D}}_n \simeq {}^a\sqrt{\infty/\mathbb{P}^1} \quad \text{and} \quad \tilde{\mathcal{D}}_{n+1} \simeq {}^a\sqrt{0/\mathbb{P}^1} \times_{\mathbb{P}^1} {}^a\sqrt{\infty/\mathbb{P}^1}.$$

Denote by $\tilde{\mathcal{D}}$ the smooth effective Cartier divisor $\tilde{\mathcal{D}}_{n+1}$.

From now on, we assume that the line bundle $\mathcal{O}_{\mathcal{X}^{can}}(\tilde{\mathcal{D}})$ is π^{can} -ample.

REMARK 3.49. As explained in the previous section, the character corresponding to the line bundle $\mathcal{O}_{\mathcal{X}^{can}}(\tilde{\mathcal{D}})$ is $\chi_{n+1}: G \hookrightarrow (\mathbb{C}^*)^{n+2} \xrightarrow{p_{n+1}} \mathbb{C}^*$ (the coordinates of $(\mathbb{C}^*)^{n+2}$ are $\lambda_0, \dots, \lambda_{n+1}$). By the π^{can} -ampleness hypothesis on $\mathcal{O}_{\mathcal{X}^{can}}(\tilde{\mathcal{D}})$ we have that the composition of the inclusion of μ_{a_0} into G and χ_{n+1} is injective and the same holds for μ_{a_∞} . We shall use this fact later on. \triangle

Let k be a positive integer and denote by \mathcal{X} the root stack $\sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}^{can}}$. It is a two-dimensional toric orbifold with coarse moduli scheme X . As we saw in the previous section, the structure morphism $\pi: \mathcal{X} \rightarrow X$ factorizes as in Corollary 1.53, and \mathcal{X} is isomorphic to the global quotient $[\tilde{Z}/\tilde{G}]$, where \tilde{Z} and \tilde{G} are defined as in Lemma 1.61, with $m = n + 2$,

$k_i = 1$ for $i = 0, \dots, n$ and $k_{n+1} = k$. Since $n+1$ of the k_i 's are 1, \tilde{Z} and \tilde{G} fit into the cartesian diagrams

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\quad} & \mathbb{A}^1 \\ \downarrow & \square & \downarrow (-)^k \\ Z & \xrightarrow{z_{n+1}} & \mathbb{A}^1 \end{array} \quad \begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\chi}_{n+1}} & \mathbb{C}^* \\ \downarrow \varphi & \square & \downarrow (-)^k \\ G & \xrightarrow{\chi_{n+1}} & \mathbb{C}^* \end{array} .$$

The action of \tilde{G} on \tilde{Z} is given by

$$(g, \lambda) \cdot (z, x) := (gz, \lambda x) ,$$

for any $(g, \lambda) \in \tilde{G}$ and $(z, x) \in \tilde{Z}$.

The effective Cartier divisor $\mathcal{D} := \pi^{-1}(D)_{red}$ is an étale μ_k -gerbe over $\tilde{\mathcal{D}}$. As a global quotient \mathcal{D} is the stack $[Z \cap \{z_{n+1} = 0\}/\tilde{G}]$, where the \tilde{G} -action is given via φ , and $\ker \varphi = \{(1, \lambda) \mid \lambda^k = 1\} \simeq \mu_k$. Moreover, the line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ corresponds to the morphism $\mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ and then to the character $\tilde{\chi}_{n+1}$.

Now we check if the hypotheses of Theorem 3.45 hold for the pair $(\mathcal{X}, \mathcal{D})$. The first thing we shall prove is that the line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is π -ample. Since $\tilde{\mathcal{D}}$ is the rigidification of \mathcal{D} with respect to μ_k (cf. [39, Section 6.3]), the stabilizer group of a geometric point p of \mathcal{D} is an extension

$$1 \rightarrow \ker \varphi \rightarrow \text{Stab}(p) \rightarrow \text{Stab}(\tilde{p}) \rightarrow 1 ,$$

where $\tilde{p} := \psi(p) \in \tilde{\mathcal{D}}$. In particular, if \tilde{p} is not \tilde{p}_0 or \tilde{p}_∞ , the stabilizer group of p is $\ker \varphi$. Since the character $(\tilde{\chi}_{n+1})|_{\ker \varphi}$ is $(1, \lambda) \mapsto \lambda$, the representation of the stabilizer group at the fiber of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ at p is faithful. If $\tilde{p} = \tilde{p}_0$, denote by p_0 the corresponding geometric point in \mathcal{D} . The kernel of the character $(\tilde{\chi}_{n+1})|_{\text{Stab}(p_0)}$ is the set $\{(g, 1) \mid g \in \text{Stab}(\tilde{p}_0) \text{ and } \chi_{n+1}(g) = 1\}$. By Remark 3.49, $(\chi_{n+1})|_{\text{Stab}(\tilde{p}_0)}$ is injective, and $(\tilde{\chi}_{n+1})|_{\text{Stab}(p_0)}$ is injective as well. Hence the representation of $\text{Stab}(p_0)$ on the fiber of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ at the point p_0 is faithful. One can argue similarly for the geometric point $p_\infty \in \mathcal{D}$ such that $\psi(p_\infty) = \tilde{p}_\infty$. Thus $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is π -ample. Therefore,

$$\mathcal{G} := \oplus_{i=1}^r \mathcal{O}_{\mathcal{X}}(\mathcal{D})^{\otimes i}$$

is a generating sheaf for \mathcal{X} for any positive integer $r \geq N_{\mathcal{X}}$, where $N_{\mathcal{X}} = \max\{k a_0, k a_\infty\}$, by Proposition 1.7. We fix a positive integer a such that $r := k a \geq N_{\mathcal{X}}$.

Now we check that Condition 3.41 holds. We shall use some arguments of [19, Section 4.2.4]. The inertia stack $\mathcal{I}(\mathcal{X})$ of \mathcal{X} has only one two-dimensional component, i.e., the stack \mathcal{X} associated with the trivial stabilizer. The one-dimensional components of $\mathcal{I}(\mathcal{X})$ are $\bigsqcup_{j=1}^{k-1} \mathcal{D}$, hence $\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}$ has nontrivial one-dimensional components. On the other hand, the one-dimensional component of the inertia stack $\mathcal{I}(\mathcal{D})$ of \mathcal{D} is $\bigsqcup_{j=0}^{k-1} \mathcal{I}(\mathcal{D})^j$, where $\Pi(\mathcal{I}(\mathcal{D})^j) = \mathcal{D}$ for any $j = 0, \dots, k-1$ (here $\Pi: \mathcal{I}(\mathcal{X}) \rightarrow \mathcal{X}$ is the forgetful morphism). After fixing a primitive k -root of unity ω , we have that $\mathcal{I}(\mathcal{D})^j$ is associated with the automorphism induced by the multiplication by ω^j for $j = 0, \dots, k-1$. Thus — roughly speaking — $\mathcal{I}(\mathcal{D})^j$ consists of pairs of the form (p, ω^j) , where p is a point of \mathcal{D} .

Let us denote by $\iota: \mathcal{D} \rightarrow \mathcal{X}$ the inclusion morphism and by $\mathcal{I}(\iota): \mathcal{I}(\mathcal{D}) \setminus \mathcal{D} \rightarrow \mathcal{I}(\mathcal{X}) \setminus \mathcal{X}$ the corresponding inclusion morphism at the level of inertia stacks. Set

$$x := \left(\text{ch}^{rep}(\mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{D}}^{\oplus \text{rk}(\mathcal{E})}) \right) \text{ch}^{rep}(\mathcal{G}^{\vee}) c_1^{rep}(\pi^* \mathcal{L}) \text{td}^{rep}(\mathcal{X}) .$$

Since the integral of $x_{\neq 1}$ is zero over the zero-dimensional components of $\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}$, we have

$$\int_{\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}}^{et} x_{\neq 1} = \int_{\mathcal{I}(\mathcal{D}) \setminus \mathcal{D}}^{et} \mathcal{I}(\iota)^*(x_{\neq 1}) = \int_{\mathcal{I}(\mathcal{D}) \setminus \mathcal{D}}^{et} [\iota^* x]_{\neq 1} .$$

Now, note that $\int_{\mathcal{D}}^{et} [\iota^* x]_1 = 0$. Indeed, $[\mathcal{I}(\iota)^* \Pi^* c_1^{rep}(\mathcal{O}_X(1))]_1 = \iota^* \Pi^* c_1^{et}(\mathcal{O}_X(1))$ and the degree zero part of the difference

$$\left[\text{ch}^{rep}(\iota^* \mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{D}}^{\oplus \text{rk}(\mathcal{E})}) \right]_1 = \text{ch}^{et}(\mathcal{F}_{\mathcal{D}}) - \text{ch}^{et}(\mathcal{O}_{\mathcal{D}}^{\oplus \text{rk}(\mathcal{E})})$$

is zero. Thus we get

$$\int_{\mathcal{I}(\mathcal{D}) \setminus \mathcal{D}}^{et} [\iota^* x]_{\neq 1} = \int_{\mathcal{D}}^{rep} \iota^* x .$$

It remains to prove that the last integral is zero. Let us fix $j \in \{1, \dots, k-1\}$. By [19, Lemma 4.14] we have

$$\text{ch}^{rep}(\mathcal{G}_{|\mathcal{I}(\mathcal{D})^j}^{\vee}) = \text{ch}^{rep}(\oplus_{i=1}^r \mathcal{O}_{\mathcal{X}}(\mathcal{D})_{|\mathcal{I}(\mathcal{D})^j}^{\otimes -i}) = \sum_{i=1}^r \omega^{-ij} \text{ch}^{et}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})_{|\mathcal{I}(\mathcal{D})^j}^{\otimes -i}) ,$$

So the zero degree part of it over $\mathcal{I}(\mathcal{D})^j$ is $\sum_{i=1}^r \omega^{-ij}$. Recall that

$$\frac{1}{k} \sum_{i=0}^{k-1} \omega^{is} = \begin{cases} 0 & s \not\equiv 0 \pmod{k} , \\ 1 & s \equiv 0 \pmod{k} . \end{cases}$$

Thus

$$\sum_{i=1}^r \omega^{-ij} = \sum_{l=1}^a \sum_{i=k(l-1)}^{kl-1} \omega^{-ij} + \omega^{-rj} - 1 = \sum_{l=1}^a \sum_{i=0}^{k-1} \omega^{-ij+k(l-1)j} + 1 - 1 = a \sum_{i=0}^{k-1} \omega^{-ij} = 0 .$$

Since the zero degree part of $\text{ch}^{rep}(\iota^* \mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{D}}^{\oplus \text{rk}(\mathcal{E})})$ is zero over $\mathcal{I}(\mathcal{D})^0$ and the zero degree part of $\text{ch}^{rep}(\mathcal{G}_{|\mathcal{D}}^{\vee})$ is zero over $\mathcal{I}(\mathcal{D})^j$ for $j = 1, \dots, k-1$, the zero degree part of

$$\text{ch}^{rep}(\iota^* \mathcal{E}) - \text{ch}^{rep}(\mathcal{O}_{\mathcal{D}}^{\oplus \text{rk}(\mathcal{E})}) \text{ch}^{rep}(\mathcal{G}_{|\mathcal{D}}^{\vee})$$

is zero over $\mathcal{I}(\mathcal{D})$ and this implies that $\int_{\mathcal{D}}^{rep} \iota^* x = 0$.

As before, Theorem 3.45 implies the following result.

THEOREM 3.50. [23, Theorem 6.10] *Let X be a normal projective toric surface and D a torus-invariant rational curve which contains the singular locus $\text{sing}(X)$ of X and is a good framing divisor. Let $\pi^{can}: \mathcal{X}^{can} \rightarrow X$ be the canonical toric orbifold of X and $\tilde{\mathcal{D}}$ the smooth effective Cartier divisor $(\pi^{can})^{-1}(D)_{red}$. Assume that $\mathcal{O}_{\mathcal{X}^{can}}(\tilde{\mathcal{D}})$ is π^{can} -ample. Let $\mathcal{X} := \sqrt[k]{\tilde{\mathcal{D}}/\mathcal{X}^{can}}$, for some positive integer k , and $\mathcal{D} \subset \mathcal{X}$ the effective Cartier divisor corresponding to the morphism $\mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$. Then for any good framing sheaf $\mathcal{F}_{\mathcal{D}}$ on \mathcal{D} and any numerical polynomial $P \in \mathbb{Q}[n]$ of degree two, there exists a fine moduli space*

parameterizing isomorphism classes of $(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$ -framed sheaves on \mathcal{X} with Hilbert polynomial P , which is a quasi-projective scheme over \mathbb{C} .

CHAPTER 4

ALE spaces and root stack compactification

In this Chapter we study the geometry of the spaces we are interested in, i.e., the ALE spaces and their stacky compactifications. In Section 4.1 we give some elements of the theory of singularities on toric surfaces, and of their relations with representation theory. Then in Section 4.2 we analyze the *rational double singularity* $\mathbb{C}^2/\mathbb{Z}_k$, and describe its *minimal resolution* $X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$. In order to study gauge theories on X_k , first we compactify it to a normal toric surface \bar{X}_k . To apply the theory developed in Chapter 3, in Section 4.3 we construct the *root stack compactification* of X_k following the procedure described in Section 3.4. We study the geometry of the resulting 2-dimensional toric Deligne-Mumford orbifold \mathcal{X}_k . In the last Section (4.4) we focus our attention on the geometry of the divisors on \mathcal{X}_k , in particular the gerbe divisor \mathcal{D}_∞ which will become the relevant *framing divisor* in the next Chapter.

4.1. Singularities of toric surfaces and their resolutions

In this section we give some elements of the theory of singularities of toric surfaces. Our main reference is [33, Chapter 10]. In particular we are interested in the local structure of the singular points, the *minimal* resolution of singularities and in the connection with representation theory, in particular the McKay correspondence.

4.1.1. Singular points on toric surfaces. Let X_Σ be a toric surface, associated with a fan $\Sigma \subset N_\mathbb{Q} \cong \mathbb{Q}^2$. The minimal generators of the rays $\rho \in \Sigma(1)$ are primitive, thus form a part of a \mathbb{Z} -basis of N . Removing from X_Σ the torus-fixed points, which turn out to be the points corresponding to the 2-dimensional cones under the Orbit-Cone correspondence [33, Theorem 3.2.6], by Theorem 1.38 we obtain a smooth toric surface. Note that there is only a finite number of such points, so X_Σ has at most finitely many singular points. For a 2-dimensional cone σ , we shall denote by p_σ the corresponding fixed point. Let σ be a 2-dimensional cone and $U_\sigma \subset X_\Sigma$ the corresponding affine toric surface, whose coordinates ring is $\mathbb{C}[\sigma^\vee \cap M]$. It is an open neighborhood of p_σ . Since a 2-dimensional cone σ in $N \cong \mathbb{Z}^2$ is always simplicial, by [33, Example 1.3.20] the affine toric surface U_σ is isomorphic to a quotient \mathbb{C}^2/G , where G is a finite abelian group, and under this isomorphism p_σ corresponds to the origin of \mathbb{C}^2/G . So p_σ is a so-called *finite abelian quotient singularity*.

We introduce a *normal form* for 2-dimensional cones that makes their study easier. The proof of the following result is based on a modified division algorithm.

PROPOSITION 4.1. [33, Proposition 10.1.1] *Let $\sigma \in N_\mathbb{Q}$ be a two-dimensional strongly convex cone. There exists a basis e_1, e_2 for N such that*

$$\sigma = \text{Cone}(e_2, de_1 - ke_2),$$

where $d > 0$, $0 \leq k < d$ and $\text{GCD}(d, k) = 1$.

We call d, k the *parameters* of the cone σ , and $\{e_1, e_2\}$ a *normalized basis* for σ .

4.1.1.1. Local structure of a singular point. Let us fix a 2-dimensional cone σ . As we saw before, $U_\sigma \simeq \mathbb{C}^2/G$. Explicitly, G is the quotient of N by the sublattice N' generated by the minimal generators of the rays in σ . Here we have $N = \mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and, by using the normal form introduced above,

$$N' = \mathbb{Z}e_2 \oplus \mathbb{Z}(de_1 - ke_2) = d\mathbb{Z}e_1 \oplus \mathbb{Z}e_2.$$

Thus

$$G = N/N' \cong \mathbb{Z}_d.$$

Note that as a consequence, for singularities on toric surfaces, the finite group G is always cyclic. Now we describe the action of the group G on \mathbb{C}^2 . Let

$$\mu_d = \{\xi \in \mathbb{C} \mid \xi^d = 1\}$$

be the group of d -th roots of unity. By fixing a primitive d -th root of unity one defines an isomorphism of groups $\mu_d \cong \mathbb{Z}_d$.

PROPOSITION 4.2. [33, Proposition 10.1.2] *Let M' be the dual lattice of N' and let m_1, m_2 be dual to the minimal generators of the cone σ in N' . Using the coordinates $x = \chi^{m_1}$ and $y = \chi^{m_2}$ of \mathbb{C}^2 , the action of $\mu_d \cong N/N'$ on \mathbb{C}^2 is given by*

$$\xi \cdot (x, y) = (\xi x, \xi^k y).$$

Moreover $U_\sigma \cong \mathbb{C}^2/\mu_d$ with respect to this action.

4.1.2. Toric resolutions of singularities. Let X be a normal toric surface. Denote by X_{sing} the (possibly empty) finite set of singular points of X . First we recall the definition of *resolution of singularities*.

DEFINITION 4.3. A proper morphism $\phi: X \rightarrow Y$ is a *resolution of singularities* of X if Y is a smooth surface and ϕ is an isomorphism outside the singular locus of X :

$$\phi: Y \setminus \phi^{-1}(X_{\text{sing}}) \xrightarrow{\sim} X \setminus X_{\text{sing}}.$$

■

The problem of finding resolutions of singularities is very difficult for general varieties. For toric surfaces, this problem admits a very simple and concrete solution, of which we give now a sketch. Let σ be a nonsmooth cone in the fan Σ . By Proposition 4.1, there is a basis e_1, e_2 of N such that $\sigma = \text{Cone}(e_2, de_1 - ke_2)$ with $d > 0$, $0 \leq k < d$ and $\text{GCD}(d, k) = 1$. Consider the refinement of Σ obtained by dividing the cone σ into two new cones

$$\begin{aligned} \sigma' &= \text{Cone}(e_2, e_1) \\ \sigma'' &= \text{Cone}(e_1, de_1 - ke_2) \end{aligned}$$

with a new 1-dimensional cone $\rho = \text{Cone}(e_1)$. Note that σ' is smooth. Moreover, if we introduce the multiplicity mult of a cone in Σ minimally generated by vector v_1, \dots, v_l as the index of the sublattice $\sum_i \mathbb{Z}v_i$ in $\sum_i \mathbb{R}v_i \cap N$, we have

$$\text{mult}(\sigma'') = k < d = \text{mult}(\sigma),$$

which means that the new fan has “simpler singularities” than the old one. This observation can be made rigorous (see the proof of Theorem 10.1.10 in [33]) and using an induction on the “complexity” of the singularities, one can construct a smooth fan Σ' which is a refinement of Σ . Moreover, the induced morphism $X_{\Sigma'} \rightarrow X_{\Sigma}$ is proper by [33, Theorem 3.4.11], and it is easy to see that it is an isomorphism outside the singular locus of X_{Σ} . Thus we have the following result.

THEOREM 4.4. [33, Theorem 10.1.10] *Let X_{Σ} be a normal toric surface. There exists a smooth fan Σ' refining Σ such that the associated toric morphism $\phi: X_{\Sigma'} \rightarrow X_{\Sigma}$ is a toric resolution of singularities.*

EXAMPLE 4.5. Consider the *rational normal cone of degree d* , which is the affine toric surface U_{σ} for $\sigma = \text{Cone}(e_2, de_1 - e_2)$. We define the fan Σ obtained by inserting a new ray $\rho = \text{Cone}(e_1)$ subdividing σ into two new 2-dimensional smooth cones

$$\begin{aligned}\sigma_1 &= \text{Cone}(e_2, e_1) \\ \sigma_2 &= \text{Cone}(e_1, de_1 - e_2).\end{aligned}$$

Then X_{Σ} is a smooth toric surface. The identity map on the lattice N defines a map of fans from Σ to σ , thus there is a corresponding toric (blow-down) morphism $\phi: X_{\Sigma} \rightarrow U_{\sigma}$. Since Σ is a refinement of σ , ϕ is proper. Moreover, if p_{σ} is the torus-fixed point corresponding to the 2-dimensional cone σ , then ϕ restricts to an isomorphism

$$X_{\Sigma} \setminus \phi^{-1}(p_{\sigma}) \cong U_{\sigma} \setminus \{p_{\sigma}\} = (U_{\sigma})_{sm},$$

that is, ϕ is a toric resolution of singularities. The inverse image $E = \phi^{-1}(p_{\sigma})$ is the curve on X_{Σ} given by the closure of the orbit corresponding to the ray ρ , which means that the singular point blows up to $E \cong \mathbb{P}^1$ on the smooth surface. We call E the *exceptional divisor*. \triangle

DEFINITION 4.6. A resolution of singularities $\phi: Y \rightarrow X$ is *minimal* if for every resolution of singularities $\psi: Z \rightarrow X$, there exists a morphism $f: Z \rightarrow Y$ through which ψ factorizes. \blacksquare

It is easy to see that a minimal resolution is unique up to isomorphism, if it exists.

REMARK 4.7. In the toric framework, also the problem of constructing minimal resolutions of singularities has an easy answer: there is an algorithmic procedure ([33, Theorem 10.2.3], which we are not describing here, that, starting from a simplicial cone σ , yields a resolution of singularities $X_{\Sigma} \rightarrow U_{\sigma}$ whose exceptional fiber contains no irreducible components E with $E \cdot E = -1$. Using the theory of birational morphisms of surfaces, it is easy to show that such a resolution is minimal (see [33, Corollary 10.4.9]). We point out that this procedure, as the one we used to obtain a resolution of singularities, consists in subdividing the singular cones into smaller smooth cones. The difference is that this one is “optimized” so to obtain a minimal resolution. \triangle

4.1.3. Representation theory and McKay correspondence. Recall, from Proposition 4.2, that the group μ_d acts on \mathbb{C}^2 via the 2-dimensional linear representation

$$(29) \quad \begin{aligned} \rho: \mu_d &\rightarrow \text{GL}(2, \mathbb{C}) \\ \xi &\mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^k \end{pmatrix}. \end{aligned}$$

The complex irreducible representations of μ_d are 1-dimensional, and each one is defined by a character $\chi_j: \xi \in \mu_d \mapsto \xi^{-j} \in \mathbb{C}^*$ for $j = 0, \dots, d-1$. We choose the minus sign for convenience.

There is an induced action of μ_d on the polynomial ring $\mathbb{C}[x, y]$ given by

$$\xi \cdot (x, y) = (\xi^{-1}x, \xi^{-k}y).$$

Each monomial $x^a y^b$ spans an invariant subspace where the action of μ_d is given by the irreducible representation with character μ_j , for $j \equiv a + kb \pmod{d}$; thus j is the weight of the monomial $x^a y^b$. Define I to be the ideal of $\mu_d \cong \{(\xi, \xi^k) \in \mathbb{C}^2 \mid \xi^d = 1\}$ as a subvariety of \mathbb{C}^2 (see [33, Section 10.3]). Then I is invariant, and the μ_k -action descends to the quotient $\mathbb{C}[x, y]/I$, which becomes a representation of μ_d . It can be shown ([33, Formula 10.3.4]) that $\mathbb{C}[x, y]/I$ is isomorphic to the regular representation of μ_d .

Now we give a brief sketch of the so-called *McKay correspondence*. Let V_j be the irreducible representation of μ_d corresponding to the character χ_{-j} . Then it can be shown ([33, Lemma 10.3.7]) that the invariant subspace $(\mathbb{C}[x, y] \otimes_{\mathbb{C}} V_j)^{\mu_d}$ is a module over the ring of invariant $\mathbb{C}[x, y]^{\mu_d}$. We call the representation V_j *special with respect to k* if $(\mathbb{C}[x, y] \otimes_{\mathbb{C}} V_j)^{\mu_d}$ is minimally generated, as a $\mathbb{C}[x, y]^{\mu_d}$ -module, by two elements. We have the following result.

THEOREM 4.8 (McKay correspondence). [33, Theorem 10.3.10] *Let σ be a cone with parameters d, k , where $0 < k < d$ and $\text{GCD}(d, k) = 1$. There is a one-to-one correspondence between the representations of μ_d that are special with respect to k and the components of the exceptional divisor for the minimal resolution $X_{\Sigma} \rightarrow U_{\sigma}$.*

4.2. Minimal resolution of $\mathbb{C}^2/\mathbb{Z}_k$

In this section we start the study of the toric variety we are interested in: the quotient $\mathbb{C}^2/\mathbb{Z}_k$. Following the previous section, we construct a minimal resolution of singularities X_k , and study its geometry, in particular its divisors and their intersection products. Then we introduce a *normal* compactification \tilde{X}_k , and again we characterize its Picard group and the intersection product on it.

Let T be the 2-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$. Let $N \simeq \mathbb{Z} \oplus \mathbb{Z}$ be the lattice of 1-parameter subgroups of T and let $M = \text{Hom}(N, \mathbb{Z})$ be the lattice of characters of T . We fix a \mathbb{Z} -basis $\{e_1, e_2\}$ of N and let $\{e_1^*, e_2^*\}$ be the dual basis of M . So e_i corresponds to the character $T_i: T \rightarrow \mathbb{C}^*$, which is the i -th projection for $i = 1, 2$.

4.2.1. Toric realization of $\mathbb{C}^2/\mathbb{Z}_k$. For any integer $i \geq 0$, define $v_i := ie_1 + (1-i)e_2$. Let us consider the 2-dimensional strongly convex rational cone $\sigma := \text{Cone}(v_0, v_k)$ for $k \geq 2$. Its dual cone σ^\vee is generated by $v_0^* = e_1^*$ and $v_k^* = (k-1)e_1^* + ke_2^*$. Hence the affine toric surface $U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ has coordinate ring

$$(30) \quad \mathbb{C}[U_\sigma] := \mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[T_1, T_1^{k-1}T_2^k].$$

On the other hand, as explained in Section 4.1.1, $\{e_1, e_2\}$ is exactly the normalized basis for σ , which means that σ is in its normal form with parameters $k, k-1$. Then

$$U_\sigma \cong \mathbb{C}^2/\mathbb{Z}_k,$$

where $\mathbb{Z}_k = N/N' \cong N/\mathbb{Z}v_0 \oplus \mathbb{Z}v_k$.

The action of \mathbb{Z}_k on \mathbb{C}^2 is given by Proposition 4.2, indeed the action of a primitive k -root of unity $\omega \in \mu_k \simeq \mathbb{Z}_k$ on \mathbb{C}^2 is given by

$$\omega \cdot (t_1, t_2) := (\omega t_1, \omega^{-1} t_2) .$$

In this way, the coordinate ring of $\mathbb{C}^2/\mathbb{Z}_k$ is

$$(31) \quad \mathbb{C}[\mathbb{C}^2/\mathbb{Z}_k] := \mathbb{C}[t_1, t_2]^{\mathbb{Z}_k} = \mathbb{C}[t_1^k, t_2^k, t_1 t_2] .$$

Since $U_\sigma \simeq \mathbb{C}^2/\mathbb{Z}_k$, the rings (30) and (31) are isomorphic by imposing

$$(32) \quad T_1 = t_1^k \quad \text{and} \quad T_2 = t_2 t_1^{1-k} .$$

Indeed, by using (32), one can prove that both rings (30) and (31) are isomorphic to the ring $\mathbb{C}[X, Y, Z]/\langle Z^k - XY \rangle$. Thus the toric surface $U_\sigma \cong \mathbb{C}^2/\mathbb{Z}_k$ may be identified with the variety $V(Z^k - XY) \subset \mathbb{C}^3$.

REMARK 4.9. For a geometric explanation of the isomorphism between these rings, we refer to [33, Proposition 1.3.18]. \triangle

Note that the origin is the unique singular point of $\mathbb{C}^2/\mathbb{Z}_k$, and is a particular case of the so-called *rational double point* or *Du Val singularity* (see [33, Definition 10.4.10]). These singularities are, from a certain point of view, the simplest ones.

4.2.2. The minimal resolution. Now we apply the procedure mentioned in Remark 4.7 (for details see [33, Section 10.2]), for constructing a resolution of singularities of U_σ . We obtain the smooth toric surface $\varphi_k: X_k \rightarrow U_\sigma$ defined by the fan $\Sigma_k \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, where

$$\begin{aligned} \Sigma_k(0) &:= \{\{0\}\}, \\ \Sigma_k(1) &:= \{\rho_i := \text{Cone}(v_i) \mid i = 0, 1, 2, \dots, k\}, \\ \Sigma_k(2) &:= \{\sigma_i := \text{Cone}(v_{i-1}, v_i) \mid i = 1, 2, \dots, k\}, \end{aligned}$$

where we denote by $\Sigma_k(j)$ the set of j -dimensional cones in Σ_k , for $j = 0, 1, 2$.

From [33, Corollary 10.4.9] it follows that X_k is a *minimal* resolution of U_σ . Note that the vectors v_i are the minimal generators of the rays ρ_i for $i = 0, 1, \dots, k$. In the following we denote by D_i the T -invariant divisor associated to the ray ρ_i for $i = 0, 1, \dots, k$; they are smooth connected projective curves of genus zero on X_k .

REMARK 4.10. Recall the McKay correspondence (Theorem 4.8). In this particular case, all the irreducible representations of μ_k are special with respect to $k-1$, so that there is a one-to-one correspondence between the irreducible representations of μ_k and the components of the exceptional divisor of the minimal resolution $\varphi_k: X_k \rightarrow U_\sigma$, which are the T -invariant divisors D_i associated to the ray ρ_i for $i = 1, \dots, k-1$ (see [33, Corollary 10.3.11]). Moreover, by [33, Formula (10.4.3)], the intersection product on $\text{Pic}(X_\Sigma)$, which is generated by D_1, \dots, D_{k-1} is given by the negative of the Cartan matrix of the root system of type A_{k-1} , namely we have

$$(D_i \cdot D_j)_{i,j=1,\dots,k-1} = \begin{pmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -2 \end{pmatrix} .$$

\triangle

For $i = 1, \dots, k$ the dual cone σ_i^\vee is generated by $v_{i-1}^* = (2-i)e_1^* + (1-i)e_2^*$ and $v_i^* = (i-1)e_1^* + ie_2^*$; hence the open torus-invariant subset $U_i := \text{Spec} \mathbb{C}[\sigma_i^\vee \cap M]$ of X_k has coordinate ring

$$(33) \quad \mathbb{C}[U_i] = \mathbb{C}[T_1^{2-i}T_2^{1-i}, T_1^{i-1}T_2^i] .$$

By using Formula (32) we can define the regular functions on U_i in terms of t_1, t_2 :

$$(34) \quad \mathbb{C}[U_i] = \mathbb{C}[t_1^{k-i+1}t_2^{1-i}, t_1^{i-k}t_2^i] .$$

Note that U_i is a smooth affine toric surface for $i = 1, \dots, k$. Moreover, the isomorphism between U_i and \mathbb{C}^2 is such that the t_1 - and t_2 -axes correspond to the divisors D_{i-1} and D_i , respectively.

After identifying the characters of T with the one-dimensional T -modules, we denote by ς_1 and ς_2 (resp. ε_1 and ε_2) the first equivariant Chern class of T_1 and T_2 (resp. t_1 and t_2). By using the explicit description (33) and (34), we give here two results which will be useful in what follows. Define the 1-dimensional T -modules

$$(35) \quad \chi_1^i(T_1, T_2) = T_1^{2-i}T_2^{1-i} \quad \text{and} \quad \chi_2^i(T_1, T_2) = T_1^{i-1}T_2^i ,$$

which, by using (32), take also the form

$$(36) \quad \chi_1^i(t_1, t_2) = t_1^{k-i+1}t_2^{1-i} \quad \text{and} \quad \chi_2^i(t_1, t_2) = t_1^{i-k}t_2^i .$$

Then we have the following results.

LEMMA 4.11. *Let $i \in \{1, 2, \dots, k\}$. Then the character of the tangent space of X_k at the torus invariant point p_i is given by*

$$(37) \quad \text{ch}_T(T_{p_i}X_k) = \chi_1^i + \chi_2^i .$$

LEMMA 4.12. *Let $i \in \{0, 1, \dots, k\}$ and $j \in \{1, 2, \dots, k\}$. Then the character of the line bundle $\mathcal{O}_{X_k}(D_i)$ at the point p_j is*

$$\text{ch}_T(\mathcal{O}_{X_k}(D_i)_{p_j}) = \begin{cases} \chi_1^j & j = i, \\ \chi_2^j & j = i+1, \\ 0 & \text{otherwise} \end{cases}$$

PROOF. By [33, Theorem 4.2.8], D_i is characterized by local data $\{(U_j, \chi^{-m_j^i})\}_{j=1, \dots, k}$, where

$$\langle m_j^i, v_l \rangle = \begin{cases} -1 & j = i, l = j \text{ or } j = i+1, l = j-1, \\ 0 & \text{otherwise} . \end{cases}$$

Since $D_i|_{U_j} = \text{div}(\chi^{-m_j^i})|_{U_j}$, we get $\text{ch}_T(\mathcal{O}_{X_k}(D_i)_{p_j}) = \chi^{-m_j^i}$. One easily finds

$$\begin{aligned} m_i^i &= (i-2)e_1^* + (i-1)e_2^* , \\ m_{i+1}^i &= -ie_1^* - (1+i)e_2^* , \\ m_j^i &= 0 \quad \text{for } j \neq i, i+1 . \end{aligned}$$

□

4.2.3. Normal compactification of X_k . Let us consider the vector $b_\infty := -v_0 - v_k = -ke_1 + (k-2)e_2$ in N . Denote by ρ_∞ the ray $\text{Cone}(b_\infty) \subset N_\mathbb{Q}$ and by v_∞ its minimal generator. For k even, $v_\infty = \frac{1}{2}b_\infty$; for k odd, $v_\infty = b_\infty$. Let $\sigma_{\infty,k+1}$ and $\sigma_{\infty,k+2}$ the two-dimensional cones $\text{Cone}(v_k, v_\infty) \subset N_\mathbb{Q}$ and $\text{Cone}(v_0, v_\infty) \subset N_\mathbb{Q}$, respectively.

Let \bar{X}_k be the normal projective toric surface defined by the fan $\bar{\Sigma}_k \subset N_\mathbb{Q}$:

$$\begin{aligned}\bar{\Sigma}_k(0) &:= \{\{0\}\}, \\ \bar{\Sigma}_k(1) &:= \{\rho_i \mid i = 0, 1, \dots, k\} \cup \{\rho_\infty\} = \Sigma_k(1) \cup \{\rho_\infty\}, \\ \bar{\Sigma}_k(2) &:= \{\sigma_i \mid i = 1, 2, \dots, k\} \cup \{\sigma_{\infty,k+1}, \sigma_{\infty,k+2}\} \\ &= \Sigma_k(2) \cup \{\sigma_{\infty,k+1}, \sigma_{\infty,k+2}\}.\end{aligned}$$

First note that $i: X_k \hookrightarrow \bar{X}_k$ as an open dense subset. We denote by D_∞ the T -invariant divisor associated to the ray ρ_∞ .

From now on, we will denote by $\tilde{k} \in \mathbb{Z}_+$ the number $k/2$ if k is even, k if k is odd.

PROPOSITION 4.13. *For any $k \geq 2$, the intersection form in $\text{Div}^T(\bar{X}_k)$ is given, on the basis of T -invariant divisors, by the matrix.*

$$(38) \quad ([D_i] \cdot [D_j])_{i,j=0,\dots,k,\infty} = \begin{pmatrix} \frac{2-k}{k} & 1 & 0 & \cdots & 0 & \frac{1}{\tilde{k}} \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \frac{2-k}{k} & \frac{1}{\tilde{k}} \\ \frac{1}{\tilde{k}} & 0 & 0 & \cdots & \frac{1}{\tilde{k}} & \frac{k}{\tilde{k}^2} \end{pmatrix}$$

PROOF. By [33, Proposition 6.4.4-(a)] we have directly $[D_\infty] \cdot [D_i] = 0$ for $i = 1, \dots, k-1$. On the other hand, by [33, Lemma 6.4.2] we get

$$[D_\infty] \cdot [D_0] = \frac{\text{mult}(\rho_0)}{\text{mult}(\sigma_{\infty,k+2})}.$$

Note that $\text{mult}(\rho_0) = 1$ and $\mathbb{Z}v_0 + \mathbb{Z}v_\infty = \tilde{k}\mathbb{Z}e_1 + \mathbb{Z}e_2$. Thus $\text{mult}(\sigma_{\infty,k+2}) = \tilde{k}$.

As before, we get

$$[D_\infty] \cdot [D_k] = \frac{\text{mult}(\rho_k)}{\text{mult}(\sigma_{\infty,k+1})}.$$

We have $\mathbb{Z}v_k + \mathbb{Z}v_\infty = \tilde{k}\mathbb{Z}e_1 + \mathbb{Z}e_2$. Thus $\text{mult}(\sigma_{\infty,k+1}) = \tilde{k}$.

Set k odd. We have $v_0 + v_\infty + v_k = 0$, hence by using [33, Proposition 6.4.4] we get

$$[D_\infty] \cdot [D_\infty] = \frac{\text{mult}(\rho_\infty)}{\text{mult}(\sigma_{\infty,k+1})} = \frac{1}{k}.$$

In the same way, using the relation $kv_1 + (2-k)v_0 + v_\infty = 0$, we get

$$[D_0] \cdot [D_0] = \frac{2-k}{k},$$

and being $kv_{k-1} + (2-k)v_k + v_\infty = 0$, we have

$$[D_k] \cdot [D_k] = \frac{2-k}{k}.$$

Using the analogous relations for k even we get

$$\begin{aligned} [D_\infty] \cdot [D_\infty] &= 2 \cdot \frac{\text{mult}(\rho_\infty)}{\text{mult}(\sigma_{\infty, k+1})} = \frac{4}{k}, \\ [D_0] \cdot [D_0] &= \frac{2-k}{k}, \\ [D_k] \cdot [D_k] &= \frac{2-k}{k}. \end{aligned}$$

By Remark 4.10, for $i, j = 1, \dots, k-1$, $[D_i] \cdot [D_j] = 1$ for $|i-j| = 1$, $[D_i] \cdot [D_i] = -2$, $[D_i] \cdot [D_j] = 0$ for $|i-j| > 1$. Moreover, by [33, Corollary 6.4.3], $[D_i] \cdot [D_j] = 1$ also for $i = 0, j = 1$ and $i = k-1, j = k$. \square

By [33, Theorem 4.2.8 and Theorem 6.3.12] we get the following result.

COROLLARY 4.14. *For any $k \geq 2$, $\tilde{k}[D_\infty]$ is a nef Cartier divisor. Similarly, $\tilde{k}[D_0]$ and $\tilde{k}[D_k]$ are Cartier divisors.*

Since for any $k \geq 2$ we have $(\tilde{k}[D_\infty])^2 = k > 0$, the divisor $\tilde{k}[D_\infty]$ is also big.

REMARK 4.15. Note that by Proposition 4.2, for any $k \geq 2$ the torus-invariant affine toric open subsets $U_{\sigma_{\infty, k+1}}$ and $U_{\sigma_{\infty, k+2}}$ are isomorphic to $\mathbb{C}^2/\mathbb{Z}_{\tilde{k}}$. In particular, for $k = 2$ one has $\tilde{k} = 1$, and hence the toric surface \bar{X}_2 is smooth; indeed, it is the second Hirzebruch surface \mathbb{F}_2 . \triangle

4.3. Stacky compactifications of X_k

Here we apply to the normal toric surface \bar{X}_k the construction described in Section 3.4. First we study the geometry of its canonical toric stack $\mathcal{X}_k^{\text{can}}$, characterizing it as a quotient stack, and focusing in particular on the divisor which contains both stacky points, namely $\tilde{\mathcal{D}}_\infty$, and on the structure of its Picard group. Then we apply the root construction of Section 1.2 along the divisor $\tilde{\mathcal{D}}_\infty$, obtaining the so-called *stacky compactification* \mathcal{X}_k . Again we study the structure of the latter as a quotient stack, its Picard group, and we introduce a particular class of line bundles, which we call *tautological line bundles*. They will be very important for the next chapter, in particular because of their behavior along the gerbe divisor \mathcal{D}_∞ . The geometry of this divisor will be accurately studied in the next Section. We conclude this Section discussing the relation between the Picard group of \mathcal{X}_k and the root lattice of type A_{k-1} .

4.3.1. Canonical stack over \bar{X}_k . Let $\pi_k^{\text{can}}: \mathcal{X}_k^{\text{can}} \rightarrow \bar{X}_k$ be the two-dimensional *canonical* toric orbifold with coarse moduli space \bar{X}_k (see Section 1.5.1), whose torus is T . The boundary divisor $\mathcal{X}_k^{\text{can}} \setminus T$ is a simple normal crossing divisor, with $k+2$ irreducible components, denoted by $\tilde{\mathcal{D}}_0, \dots, \tilde{\mathcal{D}}_k, \tilde{\mathcal{D}}_\infty$. The stacky fan of $\mathcal{X}_k^{\text{can}}$ is $\bar{\Sigma}_k^{\text{can}} = (N, \bar{\Sigma}_k, \beta^{\text{can}})$, where $\beta^{\text{can}}: \mathbb{Z}^{k+2} \rightarrow N$ is given by $\{v_0, \dots, v_k, v_\infty\}$.

By Corollary 1.55, \mathcal{X}_k^{can} is the quotient stack $[Z_{\bar{\Sigma}_k}/G_{\bar{\Sigma}_k^{can}}]$, where $Z_{\bar{\Sigma}_k}$ is the union over all cones $\sigma \in \bar{\Sigma}_k$ of the open subsets

$$Z_\sigma := \left\{ x \in \mathbb{C}^{k+2} \mid x_i \neq 0 \text{ if } \rho_i \notin \sigma \right\} \subset \mathbb{C}^{k+2}.$$

If $\rho \in \bar{\Sigma}_k(1)$ is any ray and $\sigma \in \bar{\Sigma}_k(2)$ is any two-dimensional cone containing ρ , we have $Z_{\{0\}} \subset Z_\rho \subset Z_\sigma$. Then

$$Z_{\bar{\Sigma}_k} = \bigcup_{\sigma \in \bar{\Sigma}_k(2)} Z_\sigma.$$

It follows that $Z_{\bar{\Sigma}_k}$ is the subset of \mathbb{C}^{k+2} consisting of points $x = (x_1, x_2, \dots, x_{k+2})$ such that at most two coordinates can be 0. If there are exactly two zero coordinates x_i and x_j for $1 \leq i < j \leq k+2$, these are consecutive, i.e. $1 \leq i < k+2$ and $j = i+1$, or $i = 1$ and $j = k+2$. So, $Z_{\bar{\Sigma}_k}$ is the open subset obtained from \mathbb{C}^{k+2} removing the $k+2$ codimension two linear subspaces $V(x_i, x_{i+1})$ for $i = 1, \dots, k+1$ and $V(x_1, x_{k+2})$.

Recall from Section 1.6.1.2 that the group $G_{\bar{\Sigma}_k^{can}}$ can be given as

$$G_{\bar{\Sigma}_k^{can}} = \text{Hom}_{\mathbb{Z}}(DG(\beta^{can}), \mathbb{C}^*),$$

where $DG(\beta^{can})$ is simply $\text{Coker}((\beta^{can})^* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^{k+2})$. Thus $DG(\beta^{can}) \simeq \mathbb{Z}^k$ and $G_{\bar{\Sigma}_k^{can}} \simeq (\mathbb{C}^*)^k$. The action of $G_{\bar{\Sigma}_k^{can}} \simeq (\mathbb{C}^*)^k$ on $Z_{\bar{\Sigma}_k} \subset \mathbb{C}^{k+2}$ can be computed restricting the standard action of $(\mathbb{C}^*)^{k+2}$ in the following way. By applying the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the quotient map $\mathbb{Z}^{k+2} \rightarrow DG(\beta^{can}) \simeq \mathbb{Z}^k$ we obtain the injective group morphism

$$G_{\bar{\Sigma}_k^{can}} = \text{Hom}_{\mathbb{Z}}(DG(\beta^{can}), \mathbb{C}^*) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{k+2}, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{k+2}.$$

This gives the action

$$(t_1, \dots, t_k) \cdot (z_1, \dots, z_{k+2}) = \begin{cases} \left(\prod_{i=1}^{k-1} t_i^{i-2-k} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^k z_2, t_1 z_3, \dots, t_k z_{k+2} \right) & \text{for } k \text{ odd;} \\ \left(\prod_{i=1}^{k-1} t_i^{i-1-k} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{\tilde{k}} z_2, t_1 z_3, \dots, t_k z_{k+2} \right) & \text{for } k \text{ even,} \end{cases}$$

for any $(t_1, \dots, t_k) \in G_{\bar{\Sigma}_k^{can}} \simeq (\mathbb{C}^*)^k$ and $(z_1, \dots, z_{k+2}) \in \mathbb{C}^{k+2}$.

The irreducible components $\tilde{\mathcal{D}}_i$ of $\mathcal{X}_k^{can} \setminus T$ are the effective Cartier divisors corresponding to the rays ρ_i for $i = 0, \dots, k, \infty$. Moreover, we have

$$(\pi_k^{can})^*(\mathcal{O}_{\bar{X}_k}(D_i)) \simeq \mathcal{O}_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_i)$$

for any $i = 0, \dots, k, \infty$. By Remark 1.56, the Picard group $\text{Pic}(\mathcal{X}_k^{can}) \simeq DG(\beta^{can})$ fits into the short exact sequence

$$(39) \quad 0 \longrightarrow M \longrightarrow \text{Div}_T(\bar{X}_k) \xrightarrow{(\pi_k^{can})^*} \text{Pic}(\mathcal{X}_k^{can}) \longrightarrow 0.$$

4.3.1.1. Characterization of $\tilde{\mathcal{D}}_\infty$. The effective Cartier divisor $\tilde{\mathcal{D}}_\infty \subset \mathcal{X}_k^{can}$, corresponding to the ray ρ_∞ , is a 1-dimensional toric orbifold with torus \mathbb{C}^* . Its stacky fan is, by Section 1.6.2, $\bar{\Sigma}_k^{can}/\rho_\infty := (N(\rho_\infty), \bar{\Sigma}_k/\rho_\infty, \beta^{can}(\rho_\infty))$ where $N(\rho_\infty) = N/\mathbb{Z}v_\infty \simeq \mathbb{Z}$ and the quotient fan $\bar{\Sigma}_k/\rho_\infty \subset N(\rho_\infty) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ is

$$\begin{aligned} \bar{\Sigma}_k/\rho_\infty(0) &:= \{0\}, \\ \bar{\Sigma}_k/\rho_\infty(1) &:= \{\rho'_0 := \text{Cone}(1), \rho'_\infty := \text{Cone}(-1)\}. \end{aligned}$$

Moreover, the map $\beta^{can}(\rho_\infty): \mathbb{Z}^2 \rightarrow N(\rho_\infty) \simeq \mathbb{Z}$ is defined by the multiplication by $(\tilde{k}, -\tilde{k})$. Then by Theorem 1.57, $\tilde{\mathcal{D}}_\infty$ is obtained from $D_\infty \simeq \mathbb{P}^1$ by performing a (\tilde{k}, \tilde{k}) -root stack construction on the torus fixed points $0, \infty \in D_\infty$:

$$\tilde{\mathcal{D}}_\infty \simeq {}^{(\tilde{k}, \tilde{k})}\sqrt{(0, \infty)/\mathbb{P}^1} \xrightarrow{\tilde{\pi}_k} D_\infty \simeq \mathbb{P}^1,$$

where $\tilde{\pi}_k = (\pi_k^{can})|_{\tilde{\mathcal{D}}_\infty}$.

Denote by $\tilde{p}_0, \tilde{p}_\infty$ the divisors in $\tilde{\mathcal{D}}_\infty$ corresponding to the rays ρ'_0, ρ'_∞ , respectively. These are the closed substacks $\tilde{\pi}_k^{-1}(0)_{red}$ and $\tilde{\pi}_k^{-1}(\infty)_{red}$, where 0 and ∞ are the two fixed points of $D_\infty \simeq \mathbb{P}^1$. Define also the divisor $\tilde{p} := \tilde{p}_0 - \tilde{p}_\infty$.

PROPOSITION 4.16. *The toric orbifold $\tilde{\mathcal{D}}_\infty$ is isomorphic as a quotient stack to*

$$\left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mu_{\tilde{k}}} \right],$$

where the action of $\mathbb{C}^* \times \mu_{\tilde{k}}$ on $\mathbb{C}^2 \setminus \{0\}$ is given by $(t, \omega) \cdot (z_1, z_2) = (t\omega z_1, t z_2)$ for $(t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}}$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$.

PROOF. By Proposition 1.68 and Section 1.6.1.2, $\tilde{\mathcal{D}}_\infty = [Z_{\tilde{\Sigma}_k/\rho_\infty}/G_{\tilde{\Sigma}_k^{can}/\rho_\infty}]$, where $Z_{\tilde{\Sigma}_k/\rho_\infty} = \mathbb{C}^2 \setminus \{0\}$ and $G_{\tilde{\Sigma}_k^{can}/\rho_\infty} = \text{Hom}_{\mathbb{Z}}(DG(\beta^{can}(\rho_\infty)), \mathbb{C}^*)$.

As described in Section 1.6.1.1, the abelian group $DG(\beta^{can}(\rho_\infty))$ is the cokernel of the map

$$\begin{aligned} \beta^{can}(\rho_\infty)^*: \mathbb{Z} &\rightarrow \mathbb{Z}^2, \\ m &\mapsto m\tilde{k}e_1 - m\tilde{k}e_2. \end{aligned}$$

So $DG(\beta^{can}(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$, and the quotient map $\beta^{can}(\rho_\infty)^\vee: \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$ is given by the matrix

$$(40) \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $G_{\tilde{\Sigma}_k^{can}/\rho_\infty} = \mathbb{C}^* \times \mu_{\tilde{k}}$. The action of $G_{\tilde{\Sigma}_k^{can}/\rho_\infty}$ on $\mathbb{C}^2 \setminus \{0\}$ is the restriction of the standard action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2 \setminus \{0\}$ via the immersion

$$G_{\tilde{\Sigma}_k^{can}/\rho_\infty} \simeq \mathbb{C}^* \times \mu_{\tilde{k}} \simeq \text{Hom}_{\mathbb{Z}}(DG(\beta^{can}(\rho_\infty)), \mathbb{C}^*) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*) \simeq (\mathbb{C}^*)^2$$

obtained by applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the map $\beta^{can}(\rho_\infty)^\vee$. Therefore we obtain

$$(t, \omega) \cdot (z_1, z_2) = (t\omega z_1, t z_2)$$

for $(t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}}$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$. □

Remembering the characterization of the Picard group given in Section 1.6.1.2, one can easily obtain the following result.

COROLLARY 4.17. *The Picard group $\text{Pic}(\tilde{\mathcal{D}}_\infty)$ of $\tilde{\mathcal{D}}_\infty$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$. It is generated by the line bundles $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ corresponding, respectively, to the characters*

$$(t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}} \mapsto t \in \mathbb{C}^* \quad \text{and} \quad (t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}} \mapsto \omega \in \mathbb{C}^*.$$

Now we give a geometrical interpretation of the line bundles $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$.

PROPOSITION 4.18. *The line bundle $\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty)$ is isomorphic to $\tilde{\mathcal{L}}_1$ and the line bundle $\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p})$ is isomorphic to $\tilde{\mathcal{L}}_2$.*

PROOF. Since the quotient map $\beta^{can}(\rho_\infty)^\vee: \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$ given by the matrix (40) can be interpreted as the map

$$\mathbb{Z}\rho'_0 \oplus \mathbb{Z}\rho'_\infty \rightarrow \text{Pic}(\tilde{\mathcal{D}}_\infty) ,$$

we have by Corollary 4.17

$$\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0) \simeq \tilde{\mathcal{L}}_1 \otimes \tilde{\mathcal{L}}_2 \quad \text{and} \quad \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty) \simeq \tilde{\mathcal{L}}_1 .$$

Thus

$$\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}) \simeq \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0) \otimes \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(-\tilde{p}_\infty) \simeq \tilde{\mathcal{L}}_2 .$$

□

The following result will be useful later.

LEMMA 4.19. *The line bundle $\mathcal{O}_{\mathcal{X}_k^{can}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty}}$ assumes the following form with respect to the generators $\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty)$ and $\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p})$ of $\text{Pic}(\tilde{\mathcal{D}}_\infty)$:*

$$(41) \quad \mathcal{O}_{\mathcal{X}_k^{can}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty}} \simeq \begin{cases} \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty)^{\otimes 2} \otimes \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}) & \text{for } k \text{ even} , \\ \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty) \otimes \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p})^{\otimes \frac{k+1}{2}} & \text{for } k \text{ odd} . \end{cases}$$

Moreover, for the line bundles $\mathcal{O}_{\mathcal{X}_k^{can}(\tilde{\mathcal{D}}_0)}$ and $\mathcal{O}_{\mathcal{X}_k^{can}(\tilde{\mathcal{D}}_k)}$, we have

$$\mathcal{O}_{\mathcal{X}_k^{can}(\tilde{\mathcal{D}}_0)|_{\tilde{\mathcal{D}}_\infty}} \simeq \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0) \quad \text{and} \quad \mathcal{O}_{\mathcal{X}_k^{can}(\tilde{\mathcal{D}}_k)|_{\tilde{\mathcal{D}}_\infty}} \simeq \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty) .$$

PROOF. By Proposition 1.68 and the proof given in [61, Section 5.1], it suffices to apply Gale duals to the following commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^{k+2} & \longrightarrow & \mathbb{Z}^{k-1} \longrightarrow 0 \\ & & \downarrow \tilde{\beta}^{can} & & \downarrow \beta^{can} & & \downarrow \\ 0 & \longrightarrow & N & \xrightarrow{=} & N & \longrightarrow & 0 \longrightarrow 0, \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \longrightarrow 0 \\ & & \downarrow \beta_{\rho_\infty}^{can} & & \downarrow \tilde{\beta}^{can} & & \downarrow \beta^{can}(\rho_\infty) \\ 0 & \longrightarrow & N_{\rho_\infty} & \longrightarrow & N & \longrightarrow & N(\rho_\infty) \longrightarrow 0, \end{array}$$

where $\tilde{\beta}^{can}: \mathbb{Z}^3 \rightarrow N$ is the restriction of $\beta^{can}: \mathbb{Z}^{k+2} \rightarrow N$ to the subgroup $\mathbb{Z}^3 \subset \mathbb{Z}^{k+2}$ generated by the rays $\rho_0, \rho_k, \rho_\infty$. Since N_{ρ_∞} is the subgroup of N generated by v_∞ , the map $\beta_{\rho_\infty}^{can}: \mathbb{Z} \rightarrow N_{\rho_\infty}$ sends 1 to v_∞ . Then we obtain

$$(42) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & \mathbb{Z}^{k+2} & \longrightarrow & \mathbb{Z}^3 \longrightarrow 0 \\ & & \parallel & & \downarrow (\beta^{can})^\vee & & \downarrow (\tilde{\beta}^{can})^\vee \\ 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & DG(\beta^{can}) & \longrightarrow & DG(\tilde{\beta}^{can}) \longrightarrow 0, \end{array}$$

and

$$(43) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \beta^{can}(\rho_\infty)^\vee & & \downarrow (\tilde{\beta}^{can})^\vee & & \downarrow (\beta_{\rho_\infty}^{can})^\vee \\ 0 & \longrightarrow & DG(\beta^{can}(\rho_\infty)) & \xrightarrow[\simeq]{\phi} & DG(\tilde{\beta}^{can}) & \longrightarrow & DG(\beta_{\rho_\infty}^{can}) \simeq 0 \longrightarrow 0. \end{array}$$

One can explicitly compute the map $(\tilde{\beta}^{can})^\vee : \mathbb{Z}^3 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_k$ in the commutative diagram (42), obtaining

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \text{ for } k \text{ even or } \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \text{ for } k \text{ odd.}$$

Since $\beta^{can}(\rho_\infty)^\vee$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

the map ϕ in the commutative diagram (43) is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } k \text{ even, } \quad \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \text{ for } k \text{ odd.}$$

Its inverse is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } k \text{ even or } \begin{pmatrix} 1 & 0 \\ \frac{k+1}{2} & \frac{k-1}{2} \end{pmatrix} \text{ for } k \text{ odd.}$$

The restriction map $\text{Pic}(\mathcal{X}_k) \rightarrow \text{Pic}(\tilde{\mathcal{D}}_\infty)$ is given by the composition of the map $DG(\beta^{can}) \rightarrow DG(\tilde{\beta}^{can})$ in the commutative diagram (42) with the inverse of ϕ . Since $\mathcal{O}_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_\infty)$ is the image of $(0, \dots, 0, 1) \in \mathbb{Z}^{k+2}$ via the map $(\beta^{can})^\vee$, the statement follows.

For $\mathcal{O}_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_0)$ and $\mathcal{O}_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_k)$, the result follows in the same way. \square

REMARK 4.20. By the previous lemma, it is easy to see that the line bundle $\mathcal{O}_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_\infty)$ is π_k^{can} -ample. \triangle

4.3.2. Root toric stack over \bar{X}_k . Let $\mathcal{X}_k := \sqrt[k]{\tilde{\mathcal{D}}_\infty / \mathcal{X}_k^{can}} \xrightarrow{\phi_k} \mathcal{X}_k^{can}$ be the stack obtained from \mathcal{X}_k^{can} by performing a k -root construction along the divisor $\tilde{\mathcal{D}}_\infty$ (see Definition 1.34). By Theorem 1.57, it is a 2-dimensional toric orbifold with coarse moduli space $\pi_k = \pi_k^{can} \circ \phi_k : \mathcal{X}_k \rightarrow \bar{X}_k$. Its torus is T . Moreover, its stacky fan is $\bar{\Sigma}_k := (N, \bar{\Sigma}_k, \beta)$, where $\beta : \mathbb{Z}^{k+2} \rightarrow N$ is given by $\{v_0, \dots, v_k, kv_\infty\}$.

As a quotient stack, \mathcal{X}_k is $[Z_{\bar{\Sigma}_k} / G_{\bar{\Sigma}_k}]$ where $Z_{\bar{\Sigma}_k}$ is the same as for \mathcal{X}_k^{can} , since both correspond to the fan $\bar{\Sigma}_k$. The group $G_{\bar{\Sigma}_k} = \text{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^*)$ can be computed as in the previous case: $DG(\beta)$ is $\text{Coker}(\beta^* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^{k+2})$. We find $DG(\beta) \simeq \mathbb{Z}^k$, and $G_{\bar{\Sigma}_k} \simeq (\mathbb{C}^*)^k$. By applying the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the quotient map $\mathbb{Z}^{k+2} \rightarrow DG(\beta)$ we obtain an injective morphism $G_{\bar{\Sigma}_k} \rightarrow (\mathbb{C}^*)^{k+2}$, which is

$$(t_1, \dots, t_k) \mapsto \begin{cases} \left(\prod_{i=1}^{k-1} t_i^i t_k^{2k-k^2}, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{k^2}, t_1, \dots, t_k \right) & \text{for } k \text{ odd,} \\ \left(\prod_{i=1}^{k-1} t_i^i t_k^{k-k^2}, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{k^2}, t_1, \dots, t_k \right) & \text{for } k \text{ even.} \end{cases}$$

By restricting the standard action of $(\mathbb{C}^*)^{k+2}$ on $Z_{\bar{\Sigma}_k} \subset \mathbb{C}^{k+2}$, we obtain the action of $G_{\bar{\Sigma}_k}$ on $Z_{\bar{\Sigma}_k}$.

By Theorem 1.57, the boundary divisor $\mathcal{X}_k \setminus T$ is a simple normal crossing divisor with $k+2$ irreducible components $\mathcal{D}_0, \dots, \mathcal{D}_k, \mathcal{D}_\infty$, which are the effective Cartier divisors corresponding to the rays $\rho_0, \dots, \rho_k, \rho_\infty$. Then, looking at diagram (9), one has

$$(44) \quad \pi_k^*(\mathcal{O}_{\bar{X}_k}(D_i)) \simeq \phi_k^*(\mathcal{O}_{\mathcal{D}_k^{can}}(\tilde{\mathcal{D}}_i)) \simeq \begin{cases} \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_i) & \text{for } i = 0, \dots, k, \\ \mathcal{O}_{\mathcal{X}_k}(k\mathcal{D}_\infty) & \text{for } i = \infty. \end{cases}$$

REMARK 4.21. Since \mathcal{X}_k is a quotient stack, there is a well defined integral intersection theory [35]. Moreover, since \mathcal{X}_k is smooth, its rational Chow groups are isomorphic to the rational Chow groups of \bar{X}_k via π_{k*} by [107, Proposition 6.1]. In particular,

$$\begin{aligned} \pi_{k*}(\mathcal{D}_i) &= D_i \quad \text{for } i = 0, \dots, k, \\ \pi_{k*}(\mathcal{D}_\infty) &= \frac{1}{k} D_\infty. \end{aligned}$$

△

Recall that, by Remark 4.10, the intersection products between the divisors D_i for $i = 1, \dots, k-1$ are given by minus the Cartan matrix C of the Dynkin diagram of type A_{k-1} . The matrix C is not unimodular and the inverse matrix C^{-1} is of the following form:

$$(C^{-1})^{ij} = \min(i, j) - \frac{ij}{k}.$$

We can define the classes in $\text{Pic}(\mathcal{X}_k)_{\mathbb{Q}}$

$$(45) \quad \omega_i := - \sum_{j=1}^{k-1} (C^{-1})^{ij} \mathcal{D}_j$$

for $i = 1, \dots, k-1$. Note that a priori these are not integral combination of the \mathcal{D}_i 's.

LEMMA 4.22. *The classes ω_i are integral combinations of \mathcal{D}_i for $i = 0, \dots, k$ and \mathcal{D}_∞ in $\text{Pic}(\mathcal{X}_k)$.*

PROOF. We argue along the lines of the proof of [31, Section 5.2].

Let $v_\infty = -\tilde{k}e_1 + ae_2$ be the minimal generator of ρ_∞ , then $a = \tilde{k} - 1 \in \mathbb{Z}$ if k is even, $a = k - 2$ if k is odd. Let us consider the following relations which hold in $\text{Pic}(\mathcal{X}_k)$:

$$(46) \quad 0 = \text{div}(\chi^{(1,0)}) = \mathcal{D}_1 + 2\mathcal{D}_2 + \dots + k\mathcal{D}_k - \tilde{k}k\mathcal{D}_\infty,$$

$$(47) \quad 0 = \text{div}(\chi^{(0,1)}) = \mathcal{D}_0 - \mathcal{D}_2 + \dots + (1-k)\mathcal{D}_k + ak\mathcal{D}_\infty,$$

where $\chi^{(1,0)}$ and $\chi^{(0,1)}$ are the characters of T associated to $(1,0), (0,1) \in M$, respectively. Since by definition

$$\omega_1 = - \sum_{j=1}^{k-1} \frac{(k-j)}{k} \mathcal{D}_j \quad \text{and} \quad \omega_{k-1} = - \sum_{j=1}^{k-1} \frac{j}{k} \mathcal{D}_j,$$

we get

$$\begin{aligned} \omega_1 &= \mathcal{D}_0 - \tilde{k}\mathcal{D}_\infty, \\ \omega_{k-1} &= \mathcal{D}_k - \tilde{k}\mathcal{D}_\infty. \end{aligned}$$

Moreover, for $i = 2, \dots, k-2$ we have $\omega_i = \omega_{i-1} - \omega_{k-1} - \sum_{j=i}^{k-1} \mathcal{D}_j$. This shows that the ω_i 's are actually integral combinations of the \mathcal{D}_i 's and \mathcal{D}_∞ in $\text{Pic}(\mathcal{X}_k)$. □

DEFINITION 4.23. For $i = 1, \dots, k-1$, we define the i -th tautological line bundle \mathcal{R}_i on \mathcal{X}_k as the line bundle associated with the class ω_i . ■

We choose the name “tautological” line bundles because our \mathcal{R}_i ’s plays the same role as the tautological line bundles considered by Kronheimer (see [69]). Indeed, their restriction to $X_k \subset \mathcal{X}_k$ gives exactly Kronheimer’s line bundles. Moreover, note that

$$(48) \quad \int_{\mathcal{X}_k} c_1(\mathcal{R}_i) \cdot c_1(\mathcal{R}_j) = \int_{\mathcal{X}_k} \omega_i \cdot \omega_j = -(C^{-1})^{ij} \quad \text{for } i, j = 1, \dots, k-1,$$

which is the same as in [70, Theorem A.7].

By using the relations (46) and (47), the tautological line bundles \mathcal{R}_i for $i = 1, \dots, k-1$ can be written as

$$(49) \quad \mathcal{R}_i \simeq \begin{cases} \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0 - \tilde{k}\mathcal{D}_\infty) & \text{for } i = 1, \\ \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0 - \sum_{j=2}^{i-1} (j-1)\mathcal{D}_j - (i-1)\sum_{j=i}^k \mathcal{D}_j + (i-2)\tilde{k}\mathcal{D}_\infty) & \text{for } i = 2, \dots, k-2, \\ \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_k - \tilde{k}\mathcal{D}_\infty) & \text{for } i = k-1. \end{cases}$$

PROPOSITION 4.24. The Picard group $\text{Pic}(\mathcal{X}_k)$ of \mathcal{X}_k is freely generated over \mathbb{Z} by \mathcal{R}_i for $i = 1, \dots, k-1$ and $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$.

PROOF. Recall that, from Section 1.2.3, any line bundle \mathcal{L} on \mathcal{X}_k is of the form $\phi_k^*(\mathcal{M}) \otimes \mathcal{O}_{\mathcal{X}_k}(m\mathcal{D}_\infty)$ for \mathcal{M} line bundle on $\mathcal{X}_k^{\text{can}}$ and m integer such that $0 \leq m \leq k-1$. Moreover m is unique and \mathcal{M} is unique up to isomorphism. Note that by the short exact sequence (39), \mathcal{M} is an integral combination of $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_i)$ for $i = 1, \dots, k-1$ and $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$. Therefore \mathcal{L} is an integral combination of $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_i)$ for $i = 1, \dots, k-1$ and $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$. Since the line bundles $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_j)$ for $j = 1, \dots, k-1$ are integral combinations of \mathcal{R}_i for $i = 1, \dots, k-1$, one has the assertion. □

By [17, Theorem 4.6], the images of the line bundles $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_i)$ for $i = 0, \dots, k, \infty$ in the K-theory of \mathcal{X}_k generate $K(\mathcal{X}_k)$; in addition, in $K(\mathcal{X}_k)$ these line bundles satisfy equations similar to (46) and (47). Therefore we have the following result.

COROLLARY 4.25. The images of the line bundles \mathcal{R}_i for $i = 1, \dots, k-1$ and $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ in $K(\mathcal{X}_k)$ generate $K(\mathcal{X}_k)$.

We conclude this section discussing a relation between line bundles on \mathcal{X}_k and elements in the root lattice \mathfrak{Q} of type A_{k-1} .

REMARK 4.26. As explained in [69, Section 4], the cohomology group $H^2(X_k, \mathbb{R}) \simeq \text{Pic}(X_k) \otimes_{\mathbb{Z}} \mathbb{R}$ can be identified with the real Cartan algebra \mathfrak{h} associated with the Dynkin diagram of type A_{k-1} . In this picture, $H_2(X_k, \mathbb{Z})$ with the root lattice \mathfrak{Q} of type A_{k-1} (cf. Example 2.4 and Remark 2.10). Under this correspondence, the classes $[D_1], \dots, [D_{k-1}]$ are the simple roots.

Since $\text{Pic}(\mathcal{X}_k)$ has no torsion, the map $j: \text{Pic}(\mathcal{X}_k) \rightarrow \text{Pic}(\mathcal{X}_k) \otimes_{\mathbb{Z}} \mathbb{R}$ is injective. Consider the restriction map $i^*: \text{Pic}(\mathcal{X}_k) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Pic}(X_k) \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the inclusion morphism $i: X_k \rightarrow \mathcal{X}_k$. The map i^* is surjective because of Formula (45).

Let now γ_j be the simple roots in \mathfrak{Q} , and $\gamma = \sum_{i=1}^{k-1} v_i \gamma_i$ an element of the root lattice. under the correspondence described above, this gives a linear combination of the divisors $\sum_{i=1}^{k-1} v_i [D_i]$. This fixes a unique line bundle $\mathcal{O}_{X_k}(\sum_{i=1}^{k-1} v_i D_i)$. Moreover, fixing an integer $u_\infty \in \mathbb{Z}$ and setting $\vec{u} = -C\vec{v}$, we have a line bundle $\mathcal{R}^{\vec{u}} = \bigotimes_{i=1}^{k-1} \mathcal{R}_i^{\otimes u_i} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)^{\otimes u_\infty}$ such that

$$\mathcal{R}_{|X_k}^{\vec{u}} \simeq \mathcal{O}_{X_k}(\sum_{i=1}^{k-1} v_i D_i).$$

△

4.4. Characterization of the stacky divisors

We conclude this chapter studying the geometry of the three most interesting divisors in \mathcal{X}_k , i.e., the divisors $\mathcal{D}_0, \mathcal{D}_k, \mathcal{D}_\infty$, which contain all the stacky structure. First we give a complete description of the Picard group of \mathcal{D}_∞ , looking at the relation between line bundles arising from divisors and line bundles associated with characters. Then we see how line bundles from \mathcal{X}_k restrict to \mathcal{D}_∞ , and how line bundles on $\tilde{\mathcal{D}}_\infty$ pull back on \mathcal{D}_∞ . Finally, we conclude obtaining similar results for $\mathcal{D}_0, \mathcal{D}_k$.

4.4.1. Characterization of \mathcal{D}_∞ . The divisor \mathcal{D}_∞ is isomorphic to the root stack

$$\sqrt[k]{\mathcal{O}_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_\infty)_{|\tilde{\mathcal{D}}_\infty}/\tilde{\mathcal{D}}_\infty}$$

(see Remark 1.35). So \mathcal{D}_∞ is a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T} \simeq T \times \mathcal{B}\mu_k$. Its stacky fan, by Section 1.6.2, is the quotient stacky fan $\bar{\Sigma}_k/\rho_\infty := (N(\rho_\infty), \bar{\Sigma}_k/\rho_\infty, \beta(\rho_\infty))$, where $N(\rho_\infty) = N/k\mathbb{Z}v_\infty \simeq \mathbb{Z} \oplus \mathbb{Z}_k$, the quotient fan $\bar{\Sigma}_k/\rho_\infty \subset N(\rho_\infty) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ is the same of $\tilde{\mathcal{D}}_\infty$, i.e.,

$$\begin{aligned} \bar{\Sigma}_k/\rho_\infty(0) &:= \{0\}, \\ \bar{\Sigma}_k/\rho_\infty(1) &:= \{\rho'_0 := \text{Cone}(1), \rho'_\infty := \text{Cone}(-1)\}. \end{aligned}$$

The quotient map $N \rightarrow N(\rho_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$ is given by

$$\begin{pmatrix} 1 - \tilde{k} & -\tilde{k} \\ -1 & -1 \end{pmatrix} \text{ if } k \text{ even or } \begin{pmatrix} k-2 & k \\ -\frac{k-1}{2} & -\frac{k+1}{2} \end{pmatrix} \text{ if } k \text{ odd}.$$

On the other hand, the map $\beta(\rho_\infty): \mathbb{Z}^2 \rightarrow N(\rho_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$ is given by the matrix

$$M(\beta(\rho_\infty)) = \begin{pmatrix} \tilde{k} & -\tilde{k} \\ -1 & -1 \end{pmatrix} \text{ if } k \text{ even or } M(\beta(\rho_\infty)) = \begin{pmatrix} k & -k \\ \frac{k-1}{2} & \frac{k-1}{2} \end{pmatrix} \text{ if } k \text{ odd}.$$

Note that, if we tensor $\beta(\rho_\infty)$ by \mathbb{Q} , we obtain a map $\bar{\beta}(\rho_\infty): \mathbb{Q}^2 \rightarrow N(\rho_\infty) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ given by multiplication by $(\tilde{k}, -\tilde{k})$. Thus \mathcal{D}_∞ is an *essentially trivial* gerbe with banding group $\text{Hom}_{\mathbb{Z}}(N(\rho_\infty)_{tor}, \mathbb{C}^*) \simeq \mu_k$ over its *rigidification* \mathcal{D}_∞^{rig} . By Remark 1.65 and Lemma 1.66, it follows that $\mathcal{D}_\infty^{rig} \simeq \tilde{\mathcal{D}}_\infty$, so \mathcal{D}_∞ is an essentially trivial μ_k -gerbe over $\tilde{\mathcal{D}}_\infty$. Let $\tilde{\phi}_k := (\phi_k)_{|\mathcal{D}_\infty}: \mathcal{D}_\infty \rightarrow \tilde{\mathcal{D}}_\infty$ be the μ_k -gerbe structure morphism. Moreover, $r_k := \tilde{\pi}_k \circ \tilde{\phi}_k: \mathcal{D}_\infty \rightarrow D_\infty \simeq \mathbb{P}^1$ is the projection of the coarse moduli space of \mathcal{D}_∞ .

PROPOSITION 4.27. \mathcal{D}_∞ is isomorphic as a quotient stack to

$$\left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mu_k} \right],$$

where the action is given by

$$(50) \quad (t, \omega) \cdot (z_1, z_2) = \begin{cases} (t^{\tilde{k}} \omega z_1, t^{\tilde{k}} \omega^{-1} z_2) & \text{for } k \text{ even;} \\ (t^k \omega^{\frac{k+1}{2}} z_1, t^k \omega^{\frac{k-1}{2}} z_2) & \text{for } k \text{ odd,} \end{cases}$$

for $(t, \omega) \in \mathbb{C}^* \times \mu_k$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$.

PROOF. By 1.68 and the construction in [61, Section 5.1], $\mathcal{D}_\infty = [Z_{\tilde{\Sigma}_k/\rho_\infty}/G_{\tilde{\Sigma}_k/\rho_\infty}]$, where $Z_{\tilde{\Sigma}_k/\rho_\infty} = \mathbb{C}^2 \setminus \{0\}$ is the same as for $\tilde{\mathcal{D}}_\infty$. The group is $G_{\tilde{\Sigma}_k/\rho_\infty} = \text{Hom}(DG(\beta(\rho_\infty)), \mathbb{C}^*)$. Since $N(\rho_\infty)$ has torsion, $DG(\beta(\rho_\infty))$ is obtained as in Section 1.6.1.1, namely, consider a free resolution of $N(\rho_\infty)$

$$0 \rightarrow \mathbb{Z} \xrightarrow{Q} \mathbb{Z}^2 \rightarrow N(\rho_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}_k \rightarrow 0,$$

where $Q: 1 \in \mathbb{Z} \mapsto ke_2 \in \mathbb{Z}^2$. Consider a lifting $B: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ of $\beta(\rho_\infty)$, so that B can be represented by the matrix $M(\beta(\rho_\infty))$. Define the map $[BQ]: \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ by adding the column Q to the matrix of B . Then $DG(\beta(\rho_\infty)) = \text{Coker}([BQ]^*)$ and $[BQ]^*$ is given by the matrix

$$H = \begin{pmatrix} \tilde{k} & -1 \\ -\tilde{k} & -1 \\ 0 & k \end{pmatrix} \text{ for } k \text{ even or } H = \begin{pmatrix} k & \frac{k-1}{2} \\ -k & \frac{k-1}{2} \\ 0 & k \end{pmatrix} \text{ for } k \text{ odd}.$$

In both cases, H is equivalent to

$$K = \begin{pmatrix} 1 & 0 \\ 0 & k \\ 0 & 0 \end{pmatrix},$$

this means that there exist two unimodular matrices $T \in M_3(\mathbb{Z})$, $P \in M_2(\mathbb{Z})$ such that $H = TKP$. So we have $DG(\beta(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$ and $G_{\tilde{\Sigma}_k/\rho_\infty} \simeq \mathbb{C}^* \times \mu_k$. The action of $\mathbb{C}^* \times \mu_k$ on $\mathbb{C}^2 \setminus \{0\}$ is given by composition of the standard $(\mathbb{C}^*)^2$ -action with the map $\mathbb{C}^* \times \mu_k \rightarrow (\mathbb{C}^*)^2$ obtained by applying the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the composition $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3 \rightarrow DG(\beta(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$, where the second map is the quotient map. This gives the assertion. \square

COROLLARY 4.28. *The Picard group $\text{Pic}(\mathcal{D}_\infty) \simeq DG(\beta(\rho_\infty))$ of \mathcal{D}_∞ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_k$. It is generated by the line bundles $\mathcal{L}_1, \mathcal{L}_2$ corresponding respectively to the two characters of $G_{\tilde{\Sigma}_k/\rho_\infty} \simeq \mathbb{Z} \oplus \mathbb{Z}_k$*

$$\chi_1: (t, \omega) \in \mathbb{C}^* \times \mu_k \mapsto t \in \mathbb{C}^* \text{ and } \chi_2: (t, \omega) \in \mathbb{C}^* \times \mu_k \mapsto \omega \in \mathbb{C}^*.$$

In particular $\mathcal{L}_2^{\otimes k}$ is trivial.

REMARK 4.29. By [17, Theorem 4.6], the K-theory of \mathcal{D}_∞ is generated by the images of \mathcal{L}_1 and \mathcal{L}_2 in $K(\mathcal{D}_\infty)$. \triangle

By the commutative diagram (10), we know also that $\text{Pic}(\mathcal{D}_\infty)$ fits into a commutative diagram

$$(51) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_k \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Pic}(\tilde{\mathcal{D}}_\infty) & \xrightarrow{\tilde{\phi}_k^*} & \text{Pic}(\mathcal{D}_\infty) & \longrightarrow & \mathbb{Z}_k \longrightarrow 0 \end{array}$$

where the vertical morphisms send $1 \mapsto \mathcal{O}_{\mathcal{X}_k^{\text{can}}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty}}$ and $1 \mapsto \mathcal{O}_{\mathcal{X}_k(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}}$, respectively. This means that every line bundle \mathcal{L} on \mathcal{D}_∞ can be written as $\mathcal{L} \simeq \tilde{\phi}_k^*(\mathcal{N}) \otimes \mathcal{O}_{\mathcal{X}_k(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}}^{\otimes l}$ for a line bundle \mathcal{N} on $\tilde{\mathcal{D}}_\infty$ and $0 \leq l < k$ an integer.

Now we characterize the restrictions of line bundles from \mathcal{X}_k to \mathcal{D}_∞ .

LEMMA 4.30.

$$\mathcal{O}_{\mathcal{X}_k(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}} \simeq \mathcal{L}_1.$$

Moreover, for k even

$$\mathcal{O}_{\mathcal{X}_k(\mathcal{D}_0)|_{\mathcal{D}_\infty}} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2 \quad \text{and} \quad \mathcal{O}_{\mathcal{X}_k(\mathcal{D}_k)|_{\mathcal{D}_\infty}} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes -1},$$

while for k odd

$$\mathcal{O}_{\mathcal{X}_k(\mathcal{D}_0)|_{\mathcal{D}_\infty}} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k+1}{2}} \quad \text{and} \quad \mathcal{O}_{\mathcal{X}_k(\mathcal{D}_k)|_{\mathcal{D}_\infty}} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k-1}{2}}.$$

PROOF. Here we use heavily the naturality of the Gale dual construction, and [61, Section 5.1]. Consider the following commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^{k+2} & \longrightarrow & \mathbb{Z}^{k-1} \longrightarrow 0 \\ & & \downarrow \tilde{\beta} & & \downarrow \beta & & \downarrow \\ 0 & \longrightarrow & N & \xlongequal{\quad} & N & \longrightarrow & 0 \longrightarrow 0, \end{array}$$

and

$$(52) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \longrightarrow 0 \\ & & \downarrow \beta_{\rho_\infty} & & \downarrow \tilde{\beta} & & \downarrow \beta(\rho_\infty) \\ 0 & \longrightarrow & N_{\rho_\infty} & \longrightarrow & N & \longrightarrow & N(\rho_\infty) \longrightarrow 0, \end{array}$$

where $\tilde{\beta}: \mathbb{Z}^3 \rightarrow N$ is the restriction of $\beta: \mathbb{Z}^{k+2} \rightarrow N$ to the subgroup $\mathbb{Z}^3 \subset \mathbb{Z}^{k+2}$ generated by the rays $\rho_0, \rho_k, \rho_\infty$. Since N_{ρ_∞} is generated by kv_∞ , the map β_{ρ_∞} sends 1 to kv_∞ . Taking the Gale dual in both diagrams, we obtain

$$(53) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & \mathbb{Z}^{k+2} & \longrightarrow & \mathbb{Z}^3 \longrightarrow 0 \\ & & \parallel & & \downarrow \beta^\vee & & \downarrow \tilde{\beta}^\vee \\ 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & DG(\beta) & \longrightarrow & DG(\tilde{\beta}) \longrightarrow 0, \end{array}$$

and

$$(54) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \beta(\rho_\infty)^\vee & & \downarrow \tilde{\beta}^\vee & & \downarrow \\ 0 & \longrightarrow & DG(\beta(\rho_\infty)) & \xrightarrow[\simeq]{\phi} & DG(\tilde{\beta}) & \longrightarrow & DG(\beta_{\rho_\infty}) \simeq 0 \longrightarrow 0. \end{array}$$

Explicit computations show that the map $(\tilde{\beta}^{can})^\vee : \mathbb{Z}^3 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$ in the diagram (53) is given by the matrix

$$\begin{pmatrix} \tilde{k} & \tilde{k} & 1 \\ 1 & -1 & 0 \end{pmatrix} \text{ for } k \text{ even or } \begin{pmatrix} k & k & 1 \\ \frac{k+1}{2} & \frac{k-1}{2} & 0 \end{pmatrix} \text{ for } k \text{ odd}.$$

Note that the isomorphism ϕ in diagram (54) is not uniquely determined by just imposing the commutativity of the diagram. For computing it one has to follow the construction of [61, Section 5.1]. In particular, the projective resolution of N one uses to compute the Gale dual of $\tilde{\beta} : \mathbb{Z}^3 \rightarrow DG(\tilde{\beta})$ must be compatible with the resolutions of N_{ρ_∞} and $N(\rho_\infty)$, so that the bottom exact sequence in diagram (52) gives rise to an exact sequence of complexes. Such a resolution can be constructed via the horseshoe Lemma. In this way one finds out that the map ϕ in (54) is just the identity. As in the proof of Lemma 4.19, the restriction map $\text{Pic}(\mathcal{X}_k) \rightarrow \text{Pic}(\mathcal{D}_\infty)$ is the composition of the map $DG(\beta) \rightarrow DG(\tilde{\beta})$ in the diagram (53) with the inverse of ϕ . Since $(0, \dots, 0, 1) \in \mathbb{Z}^{k+2}$ is mapped to $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty) \in \text{Pic}(\mathcal{X}_k) \simeq DG(\beta)$ via β^\vee , the line bundle $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ is mapped to \mathcal{L}_1 in $\text{Pic}(\mathcal{D}_\infty) \simeq DG(\beta(\rho_\infty))$. A similar argument proves the other two assertions. \square

It follows that the restrictions to \mathcal{D}_∞ of the tautological line bundles introduced in the previous section, give all the torsion elements in $\text{Pic}(\mathcal{D}_\infty)$.

COROLLARY 4.31. *Using Formula (49), the restrictions to \mathcal{D}_∞ of the tautological line bundles \mathcal{R}_i on \mathcal{X}_k are given by:*

$$\begin{aligned} \mathcal{R}_{i|\mathcal{D}_\infty} &\simeq \mathcal{L}_2^{\otimes i} && \text{for } k \text{ even}; \\ \mathcal{R}_{i|\mathcal{D}_\infty} &\simeq \mathcal{L}_2^{\otimes i \frac{k+1}{2}} && \text{for } k \text{ odd}. \end{aligned}$$

In particular, for $i = 1, \dots, k-1$ the line bundles $\mathcal{R}_{i|\mathcal{D}_\infty}$ are in one to one correspondence with the powers $\mathcal{L}_2^{\otimes j}$ for $j = 1, \dots, k-1$.

PROOF. First note that by the construction in the previous Lemma, for $i = 1, \dots, k-1$ we have $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}$. Thus by the definition of the \mathcal{R}_i

$$\begin{aligned} \mathcal{R}_{i|\mathcal{D}_\infty} &\simeq \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0 + (1-i)\mathcal{D}_k + (i-2)\tilde{k}\mathcal{D}_\infty)|_{\mathcal{D}_\infty} && \text{for } i = 1, \dots, k-2 \\ \mathcal{R}_{k-1|\mathcal{D}_\infty} &\simeq \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_k - \tilde{k}\mathcal{D}_\infty)|_{\mathcal{D}_\infty}. \end{aligned}$$

The result now follows from the previous Lemma. \square

Consider the divisors $p_0 := \tilde{\phi}_k^{-1}(\tilde{p}_0)_{red}$, $p_\infty := \tilde{\phi}_k^{-1}(\tilde{p}_\infty)_{red}$ corresponding to the rays ρ'_0, ρ'_∞ , respectively. From the explicit form of the Gale dual $\beta(\rho_\infty)^\vee$ obtained in the proof of Lemma 4.30, we can easily relate the line bundles on \mathcal{D}_∞ arising from the divisors and the line bundles $\mathcal{L}_1, \mathcal{L}_2$ associated with the characters. In particular we have the following result.

COROLLARY 4.32. *For k even*

$$\mathcal{O}_{\mathcal{D}_\infty}(p_0) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2 \quad \text{and} \quad \mathcal{O}_{\mathcal{D}_\infty}(p_\infty) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes -1}.$$

For k odd

$$\mathcal{O}_{\mathcal{D}_\infty}(p_0) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k+1}{2}} \quad \text{and} \quad \mathcal{O}_{\mathcal{D}_\infty}(p_\infty) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k-1}{2}}.$$

In particular, for any k we have $\mathcal{O}_{\mathcal{D}_\infty}(p_0) \simeq \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0)|_{\mathcal{D}_\infty}$ and $\mathcal{O}_{\mathcal{D}_\infty}(p_\infty) \simeq \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_k)|_{\mathcal{D}_\infty}$.

REMARK 4.33. This corollary makes it clear that for any $k > 1$, the line bundles associated with the divisors are not enough to generate the Picard group of the gerbe \mathcal{D}_∞ . This is evident if we consider the exact sequence (12), which in our case becomes

$$\mathbb{Z} \xrightarrow{\beta(\rho_\infty)^*} \mathbb{Z}^2 \xrightarrow{\beta(\rho_\infty)^\vee} \text{Pic}(\mathcal{D}_\infty) \rightarrow \text{Ext}_{\mathbb{Z}}^1(N(\rho_\infty), \mathbb{Z}) \simeq \mathbb{Z}_k \rightarrow 0;$$

indeed our previous sentence is equivalent to the fact that the cokernel of $\beta(\rho_\infty)^\vee$ is nonzero. \triangle

Finally we need to relate the Picard groups of $\tilde{\mathcal{D}}_\infty$ and \mathcal{D}_∞ , in particular making the map $\tilde{\phi}_k^*$ in diagram (51) explicit. Up to now, from the commutativity of the diagram we only know that

$$\tilde{\phi}_k^* \mathcal{O}_{\mathcal{X}_k^{\text{can}}(\tilde{\mathcal{D}}_\infty)}|_{\tilde{\mathcal{D}}_\infty} \simeq \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}^{\otimes k} \simeq \mathcal{L}_1^{\otimes k}.$$

PROPOSITION 4.34. $\tilde{\phi}_k^* \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0) \simeq \mathcal{O}_{\mathcal{D}_\infty}(p_0)$ and $\tilde{\phi}_k^* \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty) \simeq \mathcal{O}_{\mathcal{D}_\infty}(p_\infty)$. *In particular, $\tilde{\phi}_k^* \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}) \simeq \mathcal{L}_2$ generates the torsion part of the Picard group $\text{Pic}(\mathcal{D}_\infty)$ of \mathcal{D}_∞ for k odd, while for k even the line bundle $\tilde{\phi}_k^* \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}) \simeq \mathcal{L}_2^{\otimes 2}$ is not sufficient to generate all the torsion.*

PROOF. By the construction in Section 1.6.1.2 and the commutative diagram (7.21) in [39], we can give an explicit form for the map $\tilde{\phi}_k^*$ in the diagram (51) by looking at the commutative diagram:

$$(55) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \beta^{\text{can}}(\rho_\infty)^* & & \downarrow [BQ]^* & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \beta^{\text{can}}(\rho_\infty)^\vee & & \downarrow \pi_\beta & & \downarrow \\ 0 & \longrightarrow & DG(\beta^{\text{can}}(\rho_\infty)) & \xrightarrow{\tilde{\phi}_k^*} & DG(\beta(\rho_\infty)) & \longrightarrow & \mathbb{Z}_k \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0, \end{array}$$

where π_β is the quotient projection $\mathbb{Z}^3 \rightarrow \text{Coker}([BQ]^*)$ we already used to compute $\beta_{\rho_\infty}^\vee$. By the commutativity of the diagram, we get that $\tilde{\phi}_k^*: DG(\beta^{\text{can}}(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}} \rightarrow DG(\beta(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$ is represented by the matrix

$$\begin{pmatrix} \tilde{k} & 0 \\ -1 & 2 \end{pmatrix} \text{ for } k \text{ even or } \begin{pmatrix} k & 0 \\ \frac{k-1}{2} & 1 \end{pmatrix} \text{ for } k \text{ odd.}$$

The result follows by taking the images of the vectors $(1, 0), (1, 1), (0, 1)$ in $\text{Pic}(\tilde{\mathcal{D}}_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$ which correspond to the line bundles $\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0), \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty), \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p})$, respectively. \square

REMARK 4.35. Following the proof of [39, Proposition 7.20], one sees that the last short exact sequence in the diagram (55) is an element of $\text{Ext}^1(N^{\text{tor}}, \text{Pic}(\tilde{\mathcal{D}}_\infty))$, which by [39, Proposition 6.9], induces an element $[\mathcal{O}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty}] \in \text{Pic}(\tilde{\mathcal{D}}_\infty)/k\text{Pic}(\tilde{\mathcal{D}}_\infty)$. The last column of the diagram is a free (hence, projective) resolution of \mathbb{Z}_k , so we can lift the identity map of \mathbb{Z}_k to obtain a morphism of short exact sequences

$$(56) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_k \longrightarrow 0 \\ & & \tilde{f} \downarrow & & f \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Pic}(\tilde{\mathcal{D}}_\infty) & \xrightarrow{\tilde{\phi}_k^*} & \text{Pic}(\mathcal{D}_\infty) & \longrightarrow & \mathbb{Z}_k \longrightarrow 0. \end{array}$$

The choices of the liftings \tilde{f} and f are not unique. In particular the choice of \tilde{f} corresponds to a choice of a line bundle in the class $[\mathcal{O}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty}] \in \text{Pic}(\tilde{\mathcal{D}}_\infty)/k\text{Pic}(\tilde{\mathcal{D}}_\infty)$; the choice of f is equivalent to the choice of a line bundle in the class $[\mathcal{O}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}] \in \text{Pic}(\mathcal{D}_\infty)/k\text{Pic}(\mathcal{D}_\infty)$. Clearly the choices of $\mathcal{O}(\tilde{\mathcal{D}}_\infty)|_{\tilde{\mathcal{D}}_\infty}$ and $\mathcal{O}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}$ are equivalent to the choice of the maps \tilde{f} and f in a way that diagram (56) is exactly the diagram (51). \triangle

We conclude this section computing the *degree*¹ of all the line bundles on \mathcal{D}_∞ . This will be useful in the next chapter.

LEMMA 4.36. *For any line bundle $\mathcal{L} = \mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b}$ on \mathcal{D}_∞ with $a, b \in \mathbb{Z}$, we have*

$$\int_{\mathcal{D}_\infty} c_1(\mathcal{L}) = \frac{a}{k\tilde{k}^2}.$$

PROOF. First observe that for any $a, b \in \mathbb{Z}$

$$\mathcal{L}^{\otimes k\tilde{k}} \simeq (\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b})^{\otimes k\tilde{k}} \simeq \mathcal{L}_1^{\otimes ak\tilde{k}} \simeq \mathcal{O}_{\mathcal{D}_\infty}(p_\infty)^{\otimes ak}$$

Since \mathcal{D}_∞ is smooth, by [107, Proposition 6.1] the structure map $r_k: \mathcal{D}_\infty \rightarrow D_\infty$ induces an isomorphism $r_{k*}: A_*(\mathcal{D}_\infty)_\mathbb{Q} \xrightarrow{\sim} A_*(D_\infty)_\mathbb{Q} \simeq A_*(\mathbb{P}^1)_\mathbb{Q}$, therefore

$$\int_{\mathcal{D}_\infty} c_1(\mathcal{L}^{\otimes k\tilde{k}}) = \int_{\mathcal{D}_\infty} c_1(\mathcal{O}_{\mathcal{D}_\infty}(p_\infty)^{\otimes ak}) = \int_{D_\infty} r_{k*}(c_1(\mathcal{O}_{\mathcal{D}_\infty}(akp_\infty))).$$

By [107, Example 6.7], we obtain

$$r_{k*}([p_\infty]) = \frac{1}{d}[\infty],$$

where $\infty \in \mathbb{P}^1$ and d is the order of the stabilizer of the point p_∞ . By using the quotient presentation of \mathcal{D}_∞ in Proposition 4.27, one sees that the order of the stabilizer of p_0 is $k\tilde{k}$, so we have

$$(57) \quad \int_{\mathcal{D}_\infty} c_1(\mathcal{L}) = \frac{1}{k\tilde{k}} \int_{\mathcal{D}_\infty} \frac{1}{k\tilde{k}} c_1(\mathcal{O}_{\mathbb{P}^1}(ak)) = \frac{a}{k\tilde{k}^2}.$$

\square

¹We call degree of a line bundle the integral of its first Chern class.

4.4.2. Characterization of \mathcal{D}_0 and \mathcal{D}_k . By Proposition 1.68, the stacky fan of the divisor \mathcal{D}_0 is $(N(\rho_0), \bar{\Sigma}_k/\rho_0, \beta(\rho_0))$ where $N(\rho_0) = N/\mathbb{Z}v_0 \simeq \mathbb{Z}$, while the quotient fan $\bar{\Sigma}_k/\rho_0 \subset \mathbb{Q}$ is

$$\bar{\Sigma}_k/\rho_0 = \{0, \rho'_1 := \text{Cone}(1), \rho'_\infty := \text{Cone}(-1)\} ,$$

and the map $\beta(\rho_0)$ is given by

$$\begin{aligned} \beta(\rho_0): \mathbb{Z}\rho_1 \oplus \mathbb{Z}\rho_\infty &\rightarrow N(\rho_0) \simeq \mathbb{Z} , \\ (a, b) &\mapsto a - k\tilde{k}b . \end{aligned}$$

So we can realize \mathcal{D}_0 as a $k\tilde{k}$ root construction over the divisor 0 in D_0 , namely

$$\mathcal{D}_0 \simeq \sqrt[k\tilde{k}]{0/D_0} .$$

Let $\pi_0: \mathcal{D}_0 \rightarrow D_0 \simeq \mathbb{P}^1$ be the coarse moduli scheme. Then the point $p_0 \in \mathcal{D}_0$ is exactly $\pi_0^{-1}(0)_{red}$. By using the same techniques as in the proofs of Proposition 4.27 and Lemma 4.30 we obtain the following result.

PROPOSITION 4.37. *The Picard group of \mathcal{D}_0 is freely generated over \mathbb{Z} by $\mathcal{O}_{\mathcal{D}_0}(p_0)$. For any other point $p \in \mathcal{D}_0 \setminus \{p_0\}$, we have $\mathcal{O}_{\mathcal{D}_0}(p) \simeq \mathcal{O}_{\mathcal{D}_0}(k\tilde{k}p_0)$. Moreover, the following relations hold:*

$$\begin{aligned} \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0)|_{\mathcal{D}_0} &\simeq \mathcal{O}_{\mathcal{D}_0}((2-k)\tilde{k}p_0) , \\ \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_1)|_{\mathcal{D}_0} &\simeq \mathcal{O}_{\mathcal{D}_0}(k\tilde{k}p_0) , \\ \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_0} &\simeq \mathcal{O}_{\mathcal{D}_0}(p_0) . \end{aligned}$$

In the same way we can characterize the divisor \mathcal{D}_k . Again by Proposition 1.68, the stacky fan of \mathcal{D}_k is $(N(\rho_k) \simeq \mathbb{Z}, \bar{\Sigma}_k/\rho_k = \bar{\Sigma}_k/\rho_0, \beta(\rho_k))$, where

$$\begin{aligned} \beta(\rho_k): \mathbb{Z}\rho_{k-1} \oplus \mathbb{Z}\rho_\infty &\rightarrow N(\rho_k) \simeq \mathbb{Z} , \\ (a, b) &\mapsto -a + bk\tilde{k} . \end{aligned}$$

Thus in this case we have $\mathcal{D}_k \simeq \sqrt[k\tilde{k}]{\infty/D_k}$. Moreover, if we denote by $\pi_\infty: \mathcal{D}_k \rightarrow D_k$ the coarse moduli space morphism, we have $p_\infty = \pi_\infty^{-1}(\infty)_{red}$. As before we can prove the following result.

PROPOSITION 4.38. *The Picard group of \mathcal{D}_k is freely generated over \mathbb{Z} by $\mathcal{O}_{\mathcal{D}_k}(p_\infty)$. For any other point $p \in \mathcal{D}_k \setminus \{p_\infty\}$, we have $\mathcal{O}_{\mathcal{D}_k}(p) \simeq \mathcal{O}_{\mathcal{D}_k}(k\tilde{k}p_\infty)$. Moreover, the following relations hold:*

$$\begin{aligned} \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{k-1})|_{\mathcal{D}_k} &\simeq \mathcal{O}_{\mathcal{D}_k}(k\tilde{k}p_\infty) , \\ \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_k)|_{\mathcal{D}_k} &\simeq \mathcal{O}_{\mathcal{D}_k}((2-k)\tilde{k}p_\infty) , \\ \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_k} &\simeq \mathcal{O}_{\mathcal{D}_k}(p_\infty) . \end{aligned}$$

CHAPTER 5

Supersymmetric gauge theories on ALE spaces

This chapter is the central part of the thesis, with the main results and computations. In Section 5.1, we take the general theory developed in Chapter 3 and we use it on the projective toric stack \mathcal{X}_k studied in Chapter 4. We study the moduli spaces of $(\mathcal{D}_\infty, \mathcal{F})$ -framed sheaves on \mathcal{X}_k with fixed topological invariants, where the locally free sheaf \mathcal{F} is a direct sum of line bundles of the same degree. In particular we prove that these moduli spaces are smooth quasi-projective varieties, and we compute their dimension. In the rank one case, we show that the moduli space is the Hilbert schemes of points of X_k . In Section 5.2 we study these moduli spaces from the equivariant point of view. We classify the torus-fixed points and study the equivariant structure of the tangent bundle to the moduli spaces at these points, obtaining an explicit formula for its equivariant character. In the last three sections we study supersymmetric gauge theories on \mathcal{X}_k , and compute explicitly the relevant partition functions. In particular in Section 5.3 we define, along the line of Nakajima and Yoshioka [86, Section 4], the *deformed* partition function with fixed first Chern class. The rest of the section is dedicated to the explicit computation of such function, with examples for $k = 2, 3$. In Section 5.3.2 we consider the *instanton part* of the deformed partition function, obtaining a factorization formula that involves a product of the instanton part of the Nekrasov partition functions on the open affine subvarieties $U_i \simeq \mathbb{C}^2$, weighted by an *edge factor*, and the *deformed instanton part*, which factorizes as a product of classical and instanton parts of the Nekrasov partition functions on the U_i 's. We conclude the section writing down, by varying the first Chern class, the deformed partition function, its instanton part and its deformed instanton part for pure $U(r)$ -gauge theories on X_k . In the last section we consider analog partition functions with adjoint masses, obtaining again factorizations as products of the corresponding partition functions with adjoint masses on the U_i 's.

5.1. Moduli spaces of framed sheaves on \mathcal{X}_k

In this section we fix a class of framing sheaves $\mathcal{F}_\infty^{s, \vec{w}}$ on \mathcal{D}_∞ , namely direct sums of line bundles with the same degree s , and we use Theorem 3.50 to study moduli spaces $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ of $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaves on \mathcal{X}_k , with fixed rank r , first Chern class $\sum_i u_i \omega_i$ and determinant Δ . We argue along the line of [47, Section 2] to prove the smoothness of these moduli spaces, and give a formula for the dimension, which will be proved in Appendix C. Then we focus on the case $r = 1$, showing that in this case $\mathcal{M}_{1, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ is isomorphic to the Hilbert scheme of points $\text{Hilb}^\Delta(X_k)$ on X_k .

Given a vector $\vec{u} \in \mathbb{Z}^{k-1}$, we denote $\mathcal{R}^{\vec{u}} := \bigotimes_{i=1}^{k-1} \mathcal{R}_i^{\otimes u_i}$.

Let us fix $s \in \mathbb{Z}$. For $i = 0, \dots, k-1$ define the line bundles

$$\mathcal{O}_{\mathcal{D}_\infty}(s, i) = \begin{cases} \mathcal{L}_1^{\otimes s} \otimes \mathcal{L}_2^{\otimes i} & \text{for } k \text{ even ,} \\ \mathcal{L}_1^{\otimes s} \otimes \mathcal{L}_2^{\otimes i \frac{k+1}{2}} & \text{for } k \text{ odd .} \end{cases}$$

In addition, let us fix $\vec{w} := (w_0, \dots, w_{k-1}) \in \mathbb{N}^k$ and define the locally free sheaf

$$\mathcal{F}_\infty^{s, \vec{w}} := \oplus_{i=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(s, i)^{\oplus w_i} .$$

REMARK 5.1. From Section 1.1.3.1 we know that the rank of a torsion free sheaf \mathcal{E} on \mathcal{X}_k agrees with the degree zero part of its Chern character. Since $K(\mathcal{X}_k)$ and $K(\mathcal{D}_\infty)$ are both generated by line bundles (see Corollary 4.25 and Remark 4.29), the zero degree part of the Chern character is preserved under the restriction to \mathcal{D}_∞ . Hence the rank is preserved. Let $(\mathcal{E}, \phi_\mathcal{E})$ be a $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaf on \mathcal{X}_k (see Definition 3.33). Since the $\text{rk}(\mathcal{F}_\infty^{s, \vec{w}})$ is $r := \sum_{i=0}^{k-1} w_i$, also the underlying torsion free sheaf \mathcal{E} has rank r .

Moreover, the Picard group of \mathcal{X}_k is isomorphic to its second singular cohomology group with integral coefficients via the *first Chern class* map (see [59, Section 3.1.2])¹. Thus fixing the determinant line bundle of \mathcal{E} is equivalent to fixing its first Chern class. \triangle

LEMMA 5.2. *Let $(\mathcal{E}, \phi_\mathcal{E})$ be a $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaf on \mathcal{X}_k . Then the determinant $\det(\mathcal{E})$ of \mathcal{E} is of the form $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes sr}$ for $\vec{u} = (u_1, \dots, u_k) \in \mathbb{Z}^{k-1}$, where the integers u_j satisfy the condition*

$$(58) \quad \sum_{j=1}^{k-1} j u_j \equiv \sum_{i=0}^{k-1} i w_i \pmod{k} .$$

PROOF. The determinant line bundle of \mathcal{E} can be expressed as $\det(\mathcal{E}) = \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes u_\infty}$ for some integers $\vec{u} \in \mathbb{Z}^{k-1}$, $u_\infty \in \mathbb{Z}$. Since $\det(\mathcal{F}_\infty^{s, \vec{w}}) \simeq \det(\mathcal{E}|_{\mathcal{D}_\infty})$, we get

$$\otimes_{i=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(s, i)^{\otimes w_i} \simeq \mathcal{R}|_{\mathcal{D}_\infty}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}^{\otimes u_\infty} .$$

By Corollary 4.31 we have $\mathcal{R}|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(0, i)$ for $i = 1, \dots, k-1$ and $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(1, 0)$, hence we get the assertion. \square

REMARK 5.3. Let us define $\vec{v} := C^{-1}\vec{u}$. Then Formula (58) implies the following relation for $l = 1, \dots, k-1$:

$$k v_l = -l \sum_{i=0}^{k-1} i w_i \pmod{k} ,$$

Let $c \in \{0, 1, \dots, k-1\}$ be the equivalence class modulo k of $\sum_{i=0}^{k-1} i w_i$ and define $\gamma := C^{-1}\vec{e}_c - \vec{v}$ if $c > 0$, otherwise $\gamma := -\vec{v}$. Then $\gamma \in \mathbb{Z}^{\oplus k-1}$. We shall identify γ with an element in the root lattice \mathfrak{Q} as in Remark 4.26. Note that there we chose a different sign convention. \triangle

By Theorem 3.50 and Remark 4.20 there exists a *fine* moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ parameterizing isomorphism classes of $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaves $(\mathcal{E}, \phi_\mathcal{E})$ on \mathcal{X}_k , where \mathcal{E} is

¹This is a generalization of an analogous result for toric varieties (see [33, Theorem 12.3.2]).

a torsion-free sheaf of rank r , determinant $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes sr}$, where the components of $\vec{u} \in \mathbb{Z}^{k-1}$ satisfy the equation (58), and discriminant

$$\Delta := \Delta(\mathcal{E}) = \int_{\mathcal{X}_k} \left(c_2(\mathcal{E}) - \frac{r-1}{2r} c_1^2(\mathcal{E}) \right).$$

REMARK 5.4. The term *fine* means that there exists a *universal framed sheaf* $(\tilde{\mathcal{E}}, \tilde{\phi}_{\tilde{\mathcal{E}}})$ (see Remark 3.48), where $\tilde{\mathcal{E}}$ is a coherent sheaf on $\mathcal{X}_k \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$, flat over $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$, and $\tilde{\phi}_{\tilde{\mathcal{E}}}$ is a morphism of the form $\tilde{\phi}_{\tilde{\mathcal{E}}}: \tilde{\mathcal{E}} \rightarrow p_{\mathcal{X}_k}^*(\mathcal{F}_\infty^{s,\vec{w}})$, such that $(\tilde{\phi}_{\tilde{\mathcal{E}}})|_{\mathcal{D}_\infty \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})}$ is an isomorphism. In the following we shall call $\tilde{\mathcal{E}}$ a *universal sheaf*. \triangle

5.1.1. Smoothness.

LEMMA 5.5. *For any $s \in \mathbb{Z}$ and $i = 0, \dots, k-1$ the pushforward $r_{k*}(\mathcal{O}_{\mathcal{D}_\infty}(s, i))$ of $\mathcal{O}_{\mathcal{D}_\infty}(s, i)$ is*

- $r_{k*}(\mathcal{O}_{\mathcal{D}_\infty}(s, i)) = 0$ if s and i do not satisfy the following conditions:

$$\begin{aligned} s + i\tilde{k} &\equiv 0 \pmod{k} && \text{for } k \text{ even,} \\ s &\equiv 0 \pmod{k} && \text{for } k \text{ odd.} \end{aligned}$$

- otherwise,

$$r_{k*}(\mathcal{O}_{\mathcal{D}_\infty}(s, i)) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1} \left(\left\lfloor \frac{s+i\tilde{k}}{k} \right\rfloor + \left\lfloor \frac{s-i\tilde{k}}{k} \right\rfloor \right) & \text{for } k \text{ even,} \\ \mathcal{O}_{\mathbb{P}^1} \left(\left\lfloor \frac{1-k}{2} \frac{s-i\tilde{k}}{k} \right\rfloor + \left\lfloor \frac{1+k}{2} \frac{s-i\tilde{k}+i}{k} \right\rfloor \right) & \text{for } k \text{ odd.} \end{cases}$$

PROOF. Let $s \in \mathbb{Z}$ and $i = 0, \dots, k-1$. First recall that the banding group of the gerbe $\tilde{\phi}_k: \mathcal{D}_\infty \rightarrow \tilde{\mathcal{D}}_\infty$ is μ_k , which fits into the exact sequence

- for k even:

$$1 \rightarrow \mu_k \xrightarrow{i_{\text{even}}} \mathbb{C}^* \times \mu_k \xrightarrow{q_{\text{even}}} \mathbb{C}^* \times \mu_{\tilde{k}} \rightarrow 1,$$

where $i_{\text{even}}: \eta \mapsto (\eta, \eta^{\tilde{k}})$ and $q_{\text{even}}: (t, \omega) \mapsto (t^{\tilde{k}}\omega^{-1}, \omega^2)$.

- for k odd:

$$1 \rightarrow \mu_k \xrightarrow{i_{\text{odd}}} \mathbb{C}^* \times \mu_k \xrightarrow{q_{\text{odd}}} \mathbb{C}^* \times \mu_k \rightarrow 1,$$

where $i_{\text{odd}}: \eta \mapsto (\eta, 1)$ and $q_{\text{odd}}: (t, \omega) \mapsto (t^k\omega^{\frac{k-1}{2}}, \omega)$.

Moreover, any coherent sheaf on \mathcal{D}_∞ decomposes as direct sum of eigensheaves with respect to the characters of μ_k . The pushforward of $\tilde{\phi}_k$ preserves only the μ_k -invariant part of a coherent sheaf on \mathcal{D}_∞ . Thus the pushforward $(\tilde{\phi}_k)_*(\mathcal{O}_{\mathcal{D}_\infty}(s, i))$ is nonzero if and only if

$$(59) \quad s + i\tilde{k} \equiv 0 \pmod{k} \quad \text{for } k \text{ even,}$$

$$(60) \quad s \equiv 0 \pmod{k} \quad \text{for } k \text{ odd.}$$

For k even and for s and i satisfying formula (59), we get

$$\mathcal{O}_{\mathcal{D}_\infty}(s, i) \simeq \tilde{\phi}_k^* \left(\mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0)^{\otimes \frac{s+i\tilde{k}}{k}} \otimes \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty)^{\otimes \frac{s+i\tilde{k}}{k}-i} \right).$$

By the projection formula, which holds for the rigidication morphism $\tilde{\phi}_k$ (cf. [106]):

$$(\tilde{\phi}_k)_*(\mathcal{O}_{\mathcal{D}_\infty}(s, i)) \simeq \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_0)^{\otimes \frac{s+i\tilde{k}}{k}} \otimes \mathcal{O}_{\tilde{\mathcal{D}}_\infty}(\tilde{p}_\infty)^{\otimes \frac{s+i\tilde{k}}{k}-i}.$$

Recall that $\tilde{\mathcal{D}}_\infty$ is obtained from D_∞ by performing a (\tilde{k}, \tilde{k}) -root construction at the points $0, \infty \in D_\infty \simeq \mathbb{P}^1$. By using Lemma 1.36, we obtain for k even and s satisfying (59)

$$r_{k*}(\mathcal{O}_{\mathcal{D}_\infty}(s, i)) \simeq \mathcal{O}_{\mathbb{P}^1} \left(\left\lfloor \frac{s+i\tilde{k}}{k\tilde{k}} \right\rfloor + \left\lfloor \frac{s-i\tilde{k}}{k\tilde{k}} \right\rfloor \right).$$

In the same way, for k odd and for s satisfying formula (60), we get

$$r_{k*}(\mathcal{O}_{\mathcal{D}_\infty}(s, i)) \simeq \mathcal{O}_{\mathbb{P}^1} \left(\left\lfloor \frac{\frac{1-k}{2} \frac{s-ik}{k}}{k} \right\rfloor + \left\lfloor \frac{\frac{1+k}{2} \frac{s-ik}{k} + i}{k} \right\rfloor \right).$$

□

REMARK 5.6. Since

$$\begin{aligned} \left\lfloor \frac{s+i\tilde{k}}{k\tilde{k}} \right\rfloor + \left\lfloor \frac{s-i\tilde{k}}{k\tilde{k}} \right\rfloor &\leq 2 \left\lfloor \frac{s}{\tilde{k}k} \right\rfloor && \text{for } k \text{ even,} \\ \left\lfloor \frac{\frac{1-k}{2} \frac{s-ik}{k}}{k} \right\rfloor + \left\lfloor \frac{\frac{1+k}{2} \frac{s-ik}{k} + i}{k} \right\rfloor &\leq \left\lfloor \frac{s}{k^2} \right\rfloor && \text{for } k \text{ odd,} \end{aligned}$$

for any negative integer s , we have

$$H^0(\mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i)) = H^0(\mathbb{P}^1, r_{k*}(\mathcal{O}_{\mathcal{D}_\infty}(s, i))) = 0.$$

△

Thanks to Remark 5.6, we can argue exactly as in the proof of [47, Proposition 2.1] and obtain easily the following result. Note that the proof involves Serre duality, in our case for stacks, a treatment of which can be found in Appendix A.

PROPOSITION 5.7. *The Ext-group $\text{Ext}^i(\mathcal{E}', \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ vanishes for $i = 0, 2$ and for any pairs of $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaves $(\mathcal{E}, \phi_{\mathcal{E}})$ and $(\mathcal{E}', \phi_{\mathcal{E}'})$ on \mathcal{X}_k .*

By using the same arguments, we obtain also the following result.

COROLLARY 5.8. *Let $(\mathcal{E}, \phi_{\mathcal{E}})$ be a $(\mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})$ -framed sheaf on \mathcal{X}_k . Then for $i = 0, 2$*

$$H^i(\mathcal{X}_k, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = 0.$$

Using this fact we can now prove:

THEOREM 5.9. *$\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ is a smooth quasi-projective variety of dimension*

$$\dim_{\mathbb{C}}(\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})) = 2r\Delta - \sum_{j=1}^{k-1} (C^{-1})_{j,j} \vec{w}(0) \cdot \vec{w}(j),$$

where the $\vec{w}(j)$'s are the vectors $(w_j, \dots, w_{k-1}, w_1, \dots, w_{j-1})$ and C is the Cartan matrix of type A_{k-1} .

PROOF. The moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is a separated quasi-projective scheme over \mathbb{C} by Theorem 3.50. By Proposition 5.7 the Ext-group $\text{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ vanishes at all points $[(\mathcal{E}, \phi_\mathcal{E})]$ of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$, then $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is smooth at any point $[(\mathcal{E}, \phi_\mathcal{E})]$ by Corollary 3.47. Thus $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is a smooth quasi-projective variety over \mathbb{C} .

By theorem 3.29, the tangent space of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ at a point $[(\mathcal{E}, \phi_\mathcal{E})]$ is $\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$. So

$$\dim_{\mathbb{C}}(\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})) = \dim_{\mathbb{C}}(\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))) ,$$

and the latter dimension is computed in Appendix C. \square

5.1.2. The rank-one case. Let $\text{Hilb}^n(X_k)$ be the Hilbert scheme of n -points of X_k , the scheme that parameterizes 0-dimensional subschemes of X_k of length n (we will give a brief introduction to Hilbert schemes of points in Section 6.1.3). Let Z be a point of $\text{Hilb}^n(X_k)$. Then the pushforward $i_*(I_Z)$ of the ideal sheaf I_Z with respect to the inclusion morphism $i: X_k \rightarrow \mathcal{X}_k$ is a rank one torsion-free sheaf on \mathcal{X}_k with $\det(i_*(I_Z)) \simeq \mathcal{O}_{\mathcal{X}_k}$ and $\int_{\mathcal{X}_k} c_2(i_*(I_Z)) = n$. The morphism i induces an isomorphism $i: X_k \xrightarrow{\sim} \mathcal{X}_k \setminus \mathcal{D}_\infty$, hence $Z \subset X_k$ is disjoint from \mathcal{D}_∞ and therefore $i_*(I_Z)$ is locally free in a neighborhood of \mathcal{D}_∞ .

Let $\vec{u} \in \mathbb{Z}^{k-1}$ and $i \in \{0, \dots, k-1\}$ the congruence class of $\sum_{j=1}^{k-1} ju_j$ modulo k . Let $s \in \mathbb{Z}$. Then the coherent sheaf $\mathcal{E} := i_*(I_Z) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}$ is a rank one torsion-free sheaf on \mathcal{X}_k , locally free in a neighborhood of \mathcal{D}_∞ , with a framing $\phi_\mathcal{E}: \mathcal{E}|_{\mathcal{D}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathcal{D}_\infty}(s, i)$ induced canonically by the isomorphism $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(s, i)$ (cf. Corollary 4.31). So we get a $(\mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$ -framed sheaf $(\mathcal{E}, \phi_\mathcal{E})$ on \mathcal{X}_k (the line bundle $\mathcal{O}_{\mathcal{D}_\infty}(s, i)$ is equal to $\mathcal{F}_\infty^{s,\vec{w}}$ for the vector \vec{w} such that $w_i = 1$ and $w_j = 0$ for $j \neq i$). Moreover, $\det(\mathcal{E}) \simeq \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}$ and

$$\int_{\mathcal{X}_k} \text{ch}_2(\mathcal{E}) = \frac{1}{2} \int_{\mathcal{X}_k} c_1(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s})^2 - n .$$

This singles out a point $[(\mathcal{E}, \phi_\mathcal{E})]$ in $\mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$ so that an inclusion morphism

$$\iota_{(1,\vec{u},n)}: \text{Hilb}^n(X_k) \hookrightarrow \mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$$

rests defined. This argument extends straightforwardly to families of zero-dimensional subschemes of X_k of length n , so that $\iota_{(1,\vec{u},n)}$ is an inclusion morphism of fine moduli spaces.

PROPOSITION 5.10. *The inclusion morphism*

$$\iota_{(1,\vec{u},n)}: \text{Hilb}^n(X_k) \hookrightarrow \mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$$

is an isomorphism of fine moduli spaces.

PROOF. We can define an inverse morphism

$$\mathcal{I}_{(1,\vec{u},n)}: \mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i)) \rightarrow \text{Hilb}^n(X_k)$$

in the following way. Let $[(\mathcal{E}, \phi_\mathcal{E})]$ be a point in $\mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$. The torsion-free sheaf \mathcal{E} fits into the exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{\vee\vee} \rightarrow \mathcal{Q} \rightarrow 0 ,$$

where $\mathcal{E}^{\vee\vee}$ is the line bundle $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}$ and \mathcal{Q} is a zero-dimensional sheaf whose support has length n . Since \mathcal{E} is locally free in a neighborhood of \mathcal{D}_∞ , the support of \mathcal{Q} is disjoint from \mathcal{D}_∞ . So the quotient

$$\mathcal{O}_{\mathcal{X}_k} \simeq \mathcal{E}^{\vee\vee} \otimes \mathcal{O}_{\mathcal{X}_k}(-\vec{u}) \rightarrow \mathcal{Q} \otimes \mathcal{O}_{\mathcal{X}_k}(-\vec{u}) \rightarrow 0$$

defines a zero-dimensional subscheme $Z \subset \mathcal{X}_k$ of length n which is disjoint from \mathcal{D}_∞ , and the quotient

$$\mathcal{O}_{X_k} \rightarrow i^*(\mathcal{O}_Z) \rightarrow 0$$

defines a point $Z \in \text{Hilb}^n(X_k)$ and $\mathcal{E} \simeq i_*(\mathcal{I}_Z) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}$. It is easy to see that this argument can be generalized to families of framed sheaves. Moreover, $\iota_{(1,\vec{u},n)} \circ J_{(1,\vec{u},n)} = \text{id}$ and $J_{(1,\vec{u},n)} \circ \iota_{(1,\vec{u},n)} = \text{id}$. \square

REMARK 5.11. A consequence of the previous Proposition is that after fixing $i \in \{0, 1, \dots, k-1\}$, $\vec{u} \in \mathbb{Z}^{k-1}$ such that $\sum_{j=1}^{k-1} ju_j \equiv i \pmod{k}$, and $s \in \mathbb{Z}$, for any $(\mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$ -framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank one on \mathcal{X}_k , the torsion-free sheaf \mathcal{E} is isomorphic to $i_*(\mathcal{I}) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}$, where \mathcal{I} is the ideal sheaf of some zero-dimensional subscheme of X_k , and $\phi_{\mathcal{E}}$ canonically induced by the isomorphism $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(s, i)$. \triangle

Since $\iota_{(1,\vec{u},n)}$ is an isomorphism between fine moduli spaces, we obtain also an isomorphism between the corresponding universal objects. More precisely, let us denote by $\mathcal{Z} \subset \text{Hilb}^n(X_k) \times X_k$ the universal subscheme of $\text{Hilb}^n(X_k)$, whose fiber over $Z \in \text{Hilb}^n(X_k)$ is Z itself. Consider the commutative diagram

$$\begin{array}{ccc} \text{Hilb}^n(X_k) \times X_k & \xrightarrow{(\iota_{(1,\vec{u},n)}, i)} & \mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i)) \times \mathcal{X}_k \\ \downarrow & & \downarrow \\ \text{Hilb}^n(X_k) & \xrightarrow{\iota_{(1,\vec{u},n)}} & \mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i)) \end{array}$$

Then $(\iota_{(1,\vec{u},n)}, i)^*(\tilde{\mathcal{E}} \otimes p_{\mathcal{X}_k}^*(\mathcal{O}_{\mathcal{X}_k}(-\vec{u})))$ is the ideal sheaf of \mathcal{Z} and $\iota_{(1,\vec{u},n)}^*(\tilde{\phi}_{\mathcal{E}}) = 0$, where $\tilde{\mathcal{E}}$ is the universal sheaf on $\mathcal{M}_{1,\vec{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i)) \times \mathcal{X}_k$ introduced in Remark 5.4.

5.2. Torus action and tangent bundle

We start this section studying the torus-fixed points of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$. In particular we show that the framed sheaf corresponding to a fixed point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ splits as a direct sum of rank one framed sheaves, and we use the results in Section 5.1.2 to characterize the fixed points in terms of *combinatorial data*. Then we study the equivariant structure of the tangent bundle to $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$, computing in particular its equivariant Chern character at any fixed point as a sum of a *vertex contribution* and an *edge contribution*. The former one will depend on torus-fixed points of X_k , the latter on torus-invariant divisors of X_k .

Let us first recall some definitions which will be used in the combinatorial expressions which will appear below. Let $Y \subset \mathbb{N}^2$ be a Young diagram, i.e., a finite set of points $(a, b) \in \mathbb{N}^2$ which are the coordinates of the right-top vertices of cells arranged in left-justified columns, with the columns lengths weakly decreasing (each column has the same or shorter length than its predecessor). Define the *arm and leg lengths* of a box $s = (i, j) \in Y$ respectively by

$$a(s) = a_Y(s) := \lambda_i - j \quad \text{and} \quad \ell(s) = \ell_Y(s) := \lambda'_j - i,$$

where λ_i is the length of the i -th column of Y and λ'_j is the length of the j -th row of Y . We also define the *weight* $|Y|$ of a Young diagram as the number of boxes $s \in Y$. Given two Young diagrams Y, Y' , define for arbitrary equivariant parameters x, y

$$(61) \quad M_{Y, Y'}(x, y) = \sum_{s \in Y} x^{-\ell_{Y'}(s)} y^{a_Y(s)+1} + \sum_{t \in Y'} x^{\ell_Y(t)+1} y^{-a_{Y'}(t)}.$$

REMARK 5.12. For $Y = Y'$ this is nothing but the expression of the Chern character of the tangent bundle to the Hilbert scheme of $|Y|$ points of \mathbb{C}^2 , at the fixed point represented by Y (see Section 6.2). \triangle

5.2.1. Torus action and fixed points. Since from now on we shall deal with different tori, we shall denote by T_t the two-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ of \mathcal{X}_k . For any element $(\eta_1, \eta_2) \in T_t$, let $F_{(\eta_1, \eta_2)}$ be the automorphism of \mathcal{X}_k induced by the torus action. Define also T_ρ to be the maximal torus of $GL(r, \mathbb{C})$ consisting of diagonal matrices. Thus $T_\rho \simeq (\mathbb{C}^*)^r$ acts on the framing sheaf, preserving its decomposition as a direct sum of line bundles. We can define an action of the torus $T := T_t \times T_\rho$ on $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ by

$$(\eta_1, \eta_2, \vec{\rho}) \cdot [(\mathcal{E}, \phi_\mathcal{E})] := [(F_{(\eta_1, \eta_2)}^{-1})^*(\mathcal{E}), \phi'_\mathcal{E}] ,$$

where $\vec{\rho} = (\rho_1, \dots, \rho_r) \in T_\rho$ and $\phi'_\mathcal{E}$ is the composition of isomorphisms

$$\phi'_\mathcal{E} : (F_{(\eta_1, \eta_2)}^{-1})^* \mathcal{E}|_{\mathcal{D}_\infty} \xrightarrow{(F_{(\eta_1, \eta_2)}^{-1})^*(\phi_\mathcal{E})} (F_{(\eta_1, \eta_2)}^{-1})^* \mathcal{F}_\infty^{s, \vec{w}} \longrightarrow \mathcal{F}_\infty^{s, \vec{w}} \xrightarrow{\vec{\rho}} \mathcal{F}_\infty^{s, \vec{w}} ;$$

here the middle arrow is induced by the T_t -equivariant structure of any locally free sheaf $\mathcal{F}_\infty^{s, \vec{w}}$ whose restriction to \mathcal{D}_∞ is isomorphic to $\mathcal{F}_\infty^{s, \vec{w}}$.

PROPOSITION 5.13. *Let $[(\mathcal{E}, \phi_\mathcal{E})] \in \mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})^T$ be a T -fixed point. Then it decomposes as direct sum of rank-one framed sheaves*

$$(\mathcal{E}, \phi_\mathcal{E}) = \bigoplus_{\alpha=1}^r (\mathcal{E}_\alpha, \phi_\alpha) ,$$

where for $i = 0, \dots, k-1$ and $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^i w_j$ we have that

- \mathcal{E}_α is a tensor product $i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\otimes \vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}$, where \mathcal{I}_α is an ideal sheaf of zero-dimensional subscheme Z_α of X_k supported at the T_t -fixed points p_1, \dots, p_k and $\vec{u}_\alpha \in \mathbb{Z}^{k-1}$ is such that

$$(62) \quad \sum_{j=1}^{k-1} j(\vec{u}_\alpha)_j \equiv i \pmod{k} ;$$

- the framing $\phi_{\alpha|_{\mathcal{D}_\infty}} : \mathcal{E}_\alpha \xrightarrow{\sim} \mathcal{O}_{\mathcal{D}_\infty}(s, i)$ is induced canonically by the isomorphism $\mathcal{R}^{\otimes \vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes s}|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(s, i)$.

PROOF. In the following we use the same arguments as in the proof of an analogous result for framed sheaves on smooth projective surfaces [25, Proposition 3.2]. Let \mathcal{E} be a torsion-free sheaf on \mathcal{X}_k and \mathcal{K} the sheaf of rational functions on \mathcal{X}_k . Then $\mathcal{E}' := \mathcal{E} \otimes \mathcal{K}$ is a free

\mathcal{K} -module, and can be decomposed as a direct sum of rank-one \mathcal{K} -modules

$$\mathcal{E}' = \bigoplus_{\alpha=1}^r \mathcal{E}'_{\alpha}.$$

If in addition, the framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ corresponds to a point in $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})$ fixed by the T -action, the previous decomposition can be chosen so that, when restricted to \mathcal{D}_{∞} , it provides an eigenspace decomposition for the action of T . Restricting each summand to its regular sections, i.e., $\mathcal{E}_{\alpha} := \mathcal{E} \cap \mathcal{E}'_{\alpha}$, we obtain a decomposition

$$\mathcal{E} = \bigoplus_{\alpha=1}^r \mathcal{E}_{\alpha},$$

where each \mathcal{E}_{α} is a T -invariant rank-one torsion-free sheaf on \mathcal{X}_k . Moreover, the restriction $\phi_{\mathcal{E}|_{\mathcal{E}_{\alpha}}}$ gives a canonical framing to a direct summand of $\mathcal{F}_{\infty}^{s, \vec{w}}$. Reordering the indices α , for $i = 0, \dots, k-1$ and for each α such that $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^i w_j$ we have an induced framing on \mathcal{E}_{α}

$$\phi_{\alpha} := \phi_{\mathcal{E}|_{\mathcal{E}_{\alpha}}} : \mathcal{E}_{\alpha} \xrightarrow{\sim} \mathcal{O}_{\mathcal{D}_{\infty}}(s, i).$$

Thus $(\mathcal{E}_{\alpha}, \phi_{\alpha})$ is a $(\mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{D}_{\infty}}(s, i))$ -framed sheaf of rank one on \mathcal{X}_k . As explained in Remark 5.11 the torsion-free sheaf \mathcal{E}_{α} is a tensor product of an ideal sheaf \mathcal{I}_{α} of a zero-dimensional subscheme Z_{α} of length n_{α} supported on X_k and the line bundle $\mathcal{R}^{\vec{u}_{\alpha}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{\infty})^{\otimes s}$ for a vector $\vec{u}_{\alpha} \in \mathbb{Z}^{k-1}$ satisfying Formula (62) because of Lemma 5.2. Since the torsion-free sheaf \mathcal{E} is fixed by the T_t -action, Z_{α} is fixed as well. Thus it is supported at the T_t -fixed points p_1, \dots, p_k . \square

Let $[(\mathcal{E}, \phi_{\mathcal{E}})] = [\oplus_{\alpha=1}^r (\mathcal{E}_{\alpha}, \phi_{\alpha})]$ a T -fixed point in $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})$. Then

$$\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{\infty})^{\otimes sr} \simeq \det(\mathcal{E}) \simeq \otimes_{\alpha=1}^r \det(\mathcal{E}_{\alpha}) \simeq \otimes_{\alpha=1}^r (\mathcal{R}^{\vec{u}_{\alpha}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{\infty})^{\otimes s}),$$

hence $\sum_{\alpha=1}^r \vec{u}_{\alpha} = \vec{u}$. On the other hand, \mathcal{I}_{α} is an ideal sheaf of a T_t -fixed zero-dimensional subscheme Z_{α} of length n_{α} for $\alpha \in \{1, \dots, r\}$. So it is a disjoint union of zero-dimensional subschemes Z_{α}^i supported at the T_t -fixed points p_i for $i = 1, \dots, k$; each Z_{α}^i corresponds to a Young diagram Y_{α}^i (see Section 7.2). Hence Z_{α} corresponds to the set of Young diagrams $\vec{Y}_{\alpha} = \{Y_{\alpha}^i\}_{i=1, \dots, k}$ such that $\sum_{i=1}^k |Y_{\alpha}^i| = n_{\alpha}$.

Thus we can denote the point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ by the pair $(\vec{\mathbf{Y}}, \vec{\mathbf{u}})$, where

- $\vec{\mathbf{Y}} = (\vec{Y}_1, \dots, \vec{Y}_r)$ and for any $\alpha = 1, \dots, r$ the set $\vec{Y}_{\alpha} = \{Y_{\alpha}^i\}_{i=1, \dots, k}$ is such that $\sum_{i=1}^k |Y_{\alpha}^i| = n_{\alpha}$,
- $\vec{\mathbf{u}} = (\vec{u}_1, \dots, \vec{u}_r)$ and for any $\alpha = 1, \dots, r$ the vector $\vec{u}_{\alpha} = ((\vec{u}_{\alpha})_1, \dots, (\vec{u}_{\alpha})_{k-1})$ is such that $\sum_{\alpha=1}^r \vec{u}_{\alpha} = \vec{u}$.

If we set $\vec{v}_{\alpha} := C^{-1} \vec{u}_{\alpha}$ for $\alpha = 1, \dots, r$, we denote the same point by $(\vec{\mathbf{Y}}, \vec{\mathbf{v}})$, where $\vec{\mathbf{v}} = (\vec{v}_1, \dots, \vec{v}_r)$. On the other hand, if c_{α} is the equivalence class modulo k of $k(\vec{v}_{\alpha})_{k-1}$, we define $\gamma_{\alpha} := C^{-1} \vec{e}_{c_{\alpha}} - \vec{v}_{\alpha}$ if $c_{\alpha} > 0$, $\gamma_{\alpha} := -\vec{v}_{\alpha}$ otherwise, and we denote the same point by $(\vec{\mathbf{Y}}, \gamma)$, where $\gamma := (\gamma_1, \dots, \gamma_r)$. Note that for any $\alpha = 1, \dots, r$, the number c_{α} is uniquely determined by the vector \vec{w} , indeed if $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^i w_j$, we get $c_{\alpha} = i$ for $i \in \{0, 1, \dots, k-1\}$.

We shall call these the *combinatorial data* of $[(\mathcal{E}, \phi_{\mathcal{E}})]$.

REMARK 5.14. It is easy to see that, given $[(\mathcal{E}, \phi_{\mathcal{E}})]$

$$\begin{aligned} \int_{\mathcal{X}_k} \text{ch}_2(\mathcal{E}) &= \sum_{\alpha=1}^r \int_{\mathcal{X}_k} \text{ch}_2(i_*(\mathcal{I}_{\alpha}) \otimes \mathcal{R}^{\otimes \vec{u}_{\alpha}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{\infty})^{\otimes s}) \\ &= \frac{rs^2}{2k\tilde{k}^2} - \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C\vec{v}_{\alpha} - \sum_{\alpha=1}^r n_{\alpha} \in \frac{1}{2k\tilde{k}^2} \mathbb{Z}. \end{aligned}$$

Then

$$\int_{\mathcal{X}_k} c_2(\mathcal{E}) = \frac{r(r-1)s^2}{2k\tilde{k}^2} + \sum_{\alpha=1}^r n_{\alpha} - \frac{1}{2} \sum_{\alpha \neq \beta} \vec{v}_{\alpha} \cdot C\vec{v}_{\beta} \in \frac{1}{2k\tilde{k}^2} \mathbb{Z},$$

and therefore

$$\Delta = \sum_{\alpha=1}^r n_{\alpha} + \frac{r-1}{2r} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C\vec{v}_{\alpha} + \frac{1}{2r} \sum_{\alpha \neq \beta} \vec{v}_{\alpha} \cdot C\vec{v}_{\beta} \in \frac{1}{2rk} \mathbb{Z}.$$

Analog expressions can be computed by using γ and $\vec{c} := (c_1, \dots, c_r)$. As a byproduct, the previous computation shows that the discriminant of any $(\mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})$ -framed sheaf on \mathcal{X}_k is an element in $\frac{1}{2rk} \mathbb{Z}$.

Call $n := \sum_{\alpha} n_{\alpha}$. Then fixing the rank $r = 1$ gives

$$\int_{\mathcal{X}_k} c_2(\mathcal{E}) = \Delta = n \in \mathbb{Z}.$$

△

5.2.2. The tangent bundle. Consider the tangent bundle $\mathcal{T}_{\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})}$ to the moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})$. Its fiber over a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ is given, by Corollary 3.47, by

$$\left(\mathcal{T}_{\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})} \right)_{[(\mathcal{E}, \phi_{\mathcal{E}})]} = \text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_{\infty})).$$

Now we introduce the equivariant parameters of the torus T . For $j = 1, \dots, r$, let e_j be the one-dimensional T_{ρ} -module corresponding to the projection $(\mathbb{C}^*)^r \rightarrow \mathbb{C}^*$ to the j -th factor and a_j its equivariant first Chern class. Then $H_{T_{\rho}}^*(pt; \mathbb{Q}) = H^*(BT_{\rho}; \mathbb{Q}) = \mathbb{Q}[a_1, \dots, a_r]$. The parameters T_j and ς_j (resp. t_j and ε_j) for $j = 1, 2$ are introduced in Section 4.2.2. So $H_T^*(pt; \mathbb{Q}) = \mathbb{Q}[\varsigma_1, \varsigma_2, a_1, \dots, a_r]$ or, equivalently, $H_T^*(pt; \mathbb{Q}) = \mathbb{Q}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]$.

We want to compute the character

$$\text{ch}_T \left(\mathcal{T}_{\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})} \right)_{[(\mathcal{E}, \phi_{\mathcal{E}})]}$$

at a fixed point $[(\mathcal{E}, \phi_{\mathcal{E}})] \in \mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})^T$, with respect to the natural $T = T_t \times T_{\rho}$ -action introduced in Section 5.2.1. Let $(\vec{\mathbf{Y}}, \vec{\mathbf{u}})$ be the combinatorial data corresponding to the fixed point $[(\mathcal{E}, \phi_{\mathcal{E}})]$. Since the torsion-free sheaf \mathcal{E} decomposes as

$$\mathcal{E} = \bigoplus_{\alpha=1}^r \left(i_*(\mathcal{I}_{\alpha}) \otimes \mathcal{R}^{\vec{u}_{\alpha}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{\infty})^{\otimes s} \right)$$

we get

$$\begin{aligned}
\mathrm{ch}_T \left(\mathcal{T}_{\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})} \right)_{[(\mathcal{E}, \phi_{\mathcal{E}})]} &= \mathrm{ch}_T \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \\
&= - \sum_{\alpha, \beta=1}^r \mathrm{ch}_T \mathrm{Ext}^\bullet(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \\
&= - \sum_{\alpha, \beta=1}^r e_\beta e_\alpha^{-1} \mathrm{ch}_{T_t} \mathrm{Ext}^\bullet(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) .
\end{aligned}$$

Let

$$\begin{aligned}
(63) \quad L_{\alpha\beta}(t_1, t_2) &:= -\mathrm{ch}_{T_t} \mathrm{Ext}^\bullet(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \\
&= -\chi_T(\mathcal{X}_k, \mathcal{R}^{\vec{u}_\beta - \vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) ,
\end{aligned}$$

$$\begin{aligned}
(64) \quad M_{\alpha\beta}(t_1, t_2) &:= \mathrm{ch}_{T_t} \mathrm{Ext}^\bullet(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) + \\
&\quad - \mathrm{ch}_{T_t} \mathrm{Ext}^\bullet(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) ;
\end{aligned}$$

then

$$\mathrm{ch}_T \left(\mathcal{T}_{\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})} \right)_{[(\mathcal{E}, \phi_{\mathcal{E}})]} = \sum_{\alpha, \beta=1}^r e_\beta e_\alpha^{-1} (M_{\alpha\beta}(t_1, t_2) + L_{\alpha\beta}(t_1, t_2)) .$$

From now on, we denote by $\varepsilon_j^{(i)}$ the first Chern classes of the one-dimensional T -modules $\chi_j^i(t_1, t_2)$, introduced in (35) and (36). By definition the following relations hold:

$$\begin{aligned}
\varepsilon_1^{(i)} &= (k - i + 1)\varepsilon_1 + (1 - i)\varepsilon_2 , \\
\varepsilon_2^{(i)} &= (i - k)\varepsilon_1 + i\varepsilon_2 .
\end{aligned}$$

5.2.2.1. Vertex contribution.

PROPOSITION 5.15.

$$\begin{aligned}
M_{\alpha\beta}(t_1, t_2) &= \sum_{i=1}^k (\chi_1^i)^{-(v_\beta)_i + (v_\alpha)_i} (\chi_2^i)^{-(v_\beta)_{i-1} + (v_\alpha)_{i-1}} M_{Y_\alpha^i, Y_\beta^i}(\chi_1^i, \chi_2^i) \\
&= \sum_{i=1}^k (\chi_1^i)^{(\gamma_{\beta\alpha})_i - (C^{-1})^{i, c_{\beta\alpha}}} (\chi_2^i)^{(\gamma_{\beta\alpha})_{i-1} - (C^{-1})^{i-1, c_{\beta\alpha}}} M_{Y_\alpha^i, Y_\beta^i}(\chi_1^i, \chi_2^i) ,
\end{aligned}$$

where χ_1^i and χ_2^i were introduced in Section 4.2.2, we denoted $\gamma_{\beta\alpha} = \gamma_\beta - \gamma_\alpha$ and $(C^{-1})^{i, c_{\beta\alpha}} = (C^{-1})^{i, c_\beta} - (C^{-1})^{i, c_\alpha}$, and formally set $(C^{-1})^{j, 0} = 0$ for any $j = 1, \dots, k-1$.

To prove this Proposition we need some preliminary result. As described in Proposition 1.70, for any 2-dimensional cone σ in $\bar{\Sigma}_k$, one can define an open substack \mathcal{U}_σ of \mathcal{X}_k of the form $[V(\sigma)/N(\sigma)]$, where $V(\sigma) \simeq \mathbb{C}^2$ and $N(\sigma)$ is a finite abelian group acting on it. In particular, the open substack corresponding to σ_i for $i = 1, \dots, k$ is

$$\mathcal{U}_i = [V_i/N(\sigma_i)] \simeq U_i \simeq \mathbb{C}^2$$

and the open substack corresponding to $\sigma_{\infty, j}$ for $j = k+1, k+2$ is

$$\mathcal{U}_j = [V_j/N(\sigma_{\infty, j})] \simeq [\mathbb{C}^2/\mu_{k\bar{k}}] .$$

Set $\mathbf{U} = \bigsqcup_{i=1}^{k+2} V_i$. Since the morphisms $\mathbf{U} \rightarrow \bigsqcup_{i=1}^{k+2} \mathcal{U}_i$ and $\bigsqcup_{i=1}^{k+2} \mathcal{U}_i \rightarrow \mathcal{X}_k$ are étale and surjective, also the composition $\mathbf{u}: \mathbf{U} \rightarrow \mathcal{X}_k$ is étale and surjective, hence the pair (\mathbf{U}, \mathbf{u}) is an étale presentation of \mathcal{X}_k . Denote by $\mathbf{U}_\bullet \rightarrow \mathcal{X}_k$ the *strictly simplicial algebraic space* associated to the *simplicial algebraic space* obtained by taking the 0-coskeleton of (\mathbf{U}, \mathbf{u}) (cf. [92, Section 4.1]). For any $n \geq 0$

$$\mathbf{U}_n = \bigsqcup_{\substack{i_0, \dots, i_k \in \{1, \dots, k+2\} \\ i_0 < i_1 < \dots < i_n}} V_{i_0} \times_{\mathcal{X}_k} V_{i_1} \times_{\mathcal{X}_k} \dots \times_{\mathcal{X}_k} V_{i_n}.$$

By [92, Proposition 6.12], the category of coherent sheaves on \mathcal{X}_k is equivalent to the category of *simplicial* coherent sheaves on \mathbf{U}_\bullet (for the definition of simplicial coherent sheaf on a strictly simplicial algebraic space we refer to [92]).

PROOF OF PROPOSITION 5.15. As explained in [92, Section 6], one has an isomorphism between the Ext-groups of coherent sheaves on \mathcal{X}_k and the Ext-groups of simplicial coherent sheaves on \mathbf{U}_\bullet . Thus

$$\begin{aligned} & \text{Ext}^\bullet(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \text{Ext}^\bullet(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \\ & \text{Ext}^\bullet(\mathcal{R}_{|\mathbf{U}_\bullet}^{\vec{u}_\alpha}, \mathcal{R}_{|\mathbf{U}_\bullet}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathbf{U}_\bullet}) - \text{Ext}^\bullet(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}_{|\mathbf{U}_\bullet}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}_{|\mathbf{U}_\bullet}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathbf{U}_\bullet}), \end{aligned}$$

where for a coherent sheaf \mathcal{G} on \mathcal{X}_k we denote by $\mathcal{G}_{|\mathbf{U}_\bullet}$ the corresponding simplicial coherent sheaf on \mathbf{U}_\bullet (cf. [92, Proposition 6.12]).

Recall that \mathcal{I}_α and \mathcal{I}_β are ideal sheaves of zero-dimensional subschemes Z_α and Z_β supported at the T_i -fixed points p_1, \dots, p_k of X_k . So the restrictions of $i_*(\mathcal{I}_\alpha)$ and $i_*(\mathcal{I}_\beta)$ on \mathcal{U}_j are trivial for $j = k+1, k+2$. For the same reason, also the restrictions of $i_*(\mathcal{I}_\alpha)$ and $i_*(\mathcal{I}_\beta)$ on $\mathcal{U}_i \times_{\mathcal{X}_k} \mathcal{U}_l$ are trivial since $\mathcal{U}_i \times_{\mathcal{X}_k} \mathcal{U}_l \simeq U_i \cap U_l$ for $i, l = 1, \dots, k$. Then for pairwise different indices $l_1, \dots, l_i \in \{1, \dots, k+2\}$ we get

$$\begin{aligned} i_*(\mathcal{I}_\alpha)|_{\mathcal{U}_{l_1} \times_{\mathcal{X}_k} \mathcal{U}_{l_2} \times_{\mathcal{X}_k} \dots \times_{\mathcal{X}_k} \mathcal{U}_{l_i}} & \simeq \mathcal{O}_{\mathcal{X}_k|_{\mathcal{U}_{l_1} \times_{\mathcal{X}_k} \mathcal{U}_{l_2} \times_{\mathcal{X}_k} \dots \times_{\mathcal{X}_k} \mathcal{U}_{l_i}}}, \\ i_*(\mathcal{I}_\beta)|_{\mathcal{U}_{l_1} \times_{\mathcal{X}_k} \mathcal{U}_{l_2} \times_{\mathcal{X}_k} \dots \times_{\mathcal{X}_k} \mathcal{U}_{l_i}} & \simeq \mathcal{O}_{\mathcal{X}_k|_{\mathcal{U}_{l_1} \times_{\mathcal{X}_k} \mathcal{U}_{l_2} \times_{\mathcal{X}_k} \dots \times_{\mathcal{X}_k} \mathcal{U}_{l_i}}} \end{aligned}$$

unless $i = 1$ and $l_1 = 1, \dots, k$. Then $(i_*(\mathcal{I}_\alpha)|_{\mathbf{U}_\bullet})|_{\mathbf{U}_n} \simeq \mathcal{O}_{\mathbf{U}_\bullet|_{\mathbf{U}_n}}$ and $(i_*(\mathcal{I}_\beta)|_{\mathbf{U}_\bullet})|_{\mathbf{U}_n} \simeq \mathcal{O}_{\mathbf{U}_\bullet|_{\mathbf{U}_n}}$ for $n \geq 1$. So by using the local-to-global spectral sequence (which degenerates since \mathbf{U} is a disjoint union of affine spaces)

$$\begin{aligned} & \text{Ext}^\bullet(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \text{Ext}^\bullet(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \\ & = \sum_{i=1}^k \sum_{j=0}^2 (-1)^j \left(H^0(U_i, \mathcal{O}_{\alpha\beta|X_k}^j) - H^0(U_i, \mathcal{E}_{\alpha\beta|X_k}^j) \right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{O}_{\alpha\beta}^j & := \mathcal{E}xt^j(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)), \\ \mathcal{E}_{\alpha\beta}^j & := \mathcal{E}xt^j(i_*(\mathcal{I}_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, i_*(\mathcal{I}_\beta) \otimes \mathcal{R}^{\vec{u}_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)). \end{aligned}$$

By using the same arguments as in the proof of [47, Proposition 5.1], where $M_{\alpha\beta}$ is computed for framed sheaves on smooth projective toric surfaces, we get

$$M_{\alpha\beta}(t_1, t_2) = \sum_{i=1}^k \frac{\text{ch}_{T_i}(\mathcal{R}_{p_i}^{\vec{u}_\beta})}{\text{ch}_{T_i}(\mathcal{R}_{p_i}^{\vec{u}_\alpha})} M_{Y_\alpha^i, Y_\beta^i}(\chi_1^i(t_1, t_2), \chi_2^i(t_1, t_2)).$$

The computation of $\text{ch}_{T_i}(\mathcal{R}_{p_i}^{\vec{u}})$ for $i = 1, \dots, k$ and any vector \vec{u} can be done by using Lemma 4.12 and the relation (45). \square

Let $\vec{Y} = (Y_1, \dots, Y_r)$ be a vector of Young diagrams, \vec{b} a vector of length r and $\alpha, \beta \in \{1, \dots, r\}$. Define

$$m_{\alpha\beta}^{\vec{Y}}(x, y, \vec{b}) := \prod_{s \in Y_\alpha} (-\ell_{Y_\beta}(s)x + (a_{Y_\alpha}(s) + 1)y + b_\beta - b_\alpha) \prod_{t \in Y_\beta} ((\ell_{Y_\alpha}(t) + 1)x - a_{Y_\beta}(t)y + b_\beta - b_\alpha) .$$

Define also for $i = 1, \dots, k$ the vectors $\vec{Y}^i := (Y_1^i, \dots, Y_r^i)$ and

$$(65) \quad \vec{a}^{(i)} := \vec{a} - (\vec{v})_i \varepsilon_1^{(i)} - (\vec{v})_{i-1} \varepsilon_2^{(i)} ,$$

where $(\vec{v})_l := ((\vec{v}_1)_l, \dots, (\vec{v}_r)_l)$ for $l = 1, \dots, k-1$ and $(\vec{v})_0 = (\vec{v})_k = 0$.

Considering the Euler class of $\mathcal{T}_{\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})}$ instead of the Chern character, one has immediately

COROLLARY 5.16. *The “vertex” contribution to the Euler class of $\mathcal{T}_{\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})}$ is*

$$\prod_{\alpha, \beta=1}^r \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}) .$$

5.2.2.2. *Edge contribution.* Recall

$$\begin{aligned} L_{\alpha\beta}(t_1, t_2) &:= -\chi_T(\mathcal{X}_k, \mathcal{R}^{\vec{u}_\beta - \vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \\ &= -\chi_T(\mathcal{X}_k, \otimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes (u_\beta)_j - (u_\alpha)_j} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) . \end{aligned}$$

PROPOSITION 5.17. *Let $\vec{u}_{\beta\alpha} := \vec{u}_\beta - \vec{u}_\alpha$, then*

$$L_{\alpha\beta}(t_1, t_2) = \sum_{l=1}^{k-1} L_{\alpha\beta}^l(\chi_1^l(t_1, t_2), \chi_2^l(t_1, t_2)) ,$$

Explicit expressions for $L_{\alpha\beta}^l$ are given in Formulae (66) and (67) below.

For giving the explicit formulae for the $L_{\alpha\beta}^l$'s, we have to introduce some notation. Set $c \in \{0, \dots, k-1\}$ to be the equivalence class modulo k of $k(C^{-1}\vec{u}_{\beta\alpha})_{k-1}$, and define $\vec{v} := C^{-1}(\vec{u}_{\beta\alpha} - e_c)$ with the convention that $e_0 = 0$. Fix $l \in \{1, \dots, k-1\}$ and denote $d = d(l, c) := \delta_{l,c} - v_{l+1}$. For $v_l \geq 0$ we have

$$(66) \quad L_{\alpha\beta}^l(\chi_1^l, \chi_2^l) = \begin{cases} -\sum_{i=0}^{v_l-1} \sum_{j=0}^{d+2i} (\chi_1^l)^{[d/2]+i} (\chi_2^l)^j & \text{for } d \geq 0 , \\ \sum_{i=1}^{-[d/2]-1} \sum_{j=1}^{-2\{d/2\}+2i} (\chi_1^l)^{2\{d/2\}-i} (\chi_2^l)^{-j} + \\ \quad -\sum_{i=0}^{[d/2]+v_l-1} \sum_{j=0}^{2\{d/2\}+2i} (\chi_1^l)^i (\chi_2^l)^j & \text{for } 2-2v_l \leq d < 0 , \\ \sum_{i=0}^{v_l-1} \sum_{j=1}^{-d-2i-1} (\chi_1^l)^{-[d/2]+i} (\chi_2^l)^{-j} & \text{for } d < 2-2v_l . \end{cases}$$

For $v_l < 0$ we have similar expressions:

$$(67) \quad L_{\alpha\beta}^l(\chi_1^l, \chi_2^l) = \begin{cases} \sum_{i=1}^{-v_l} \sum_{j=1}^{-d+2i-1} (\chi_1^l)^{-\lfloor -d/2 \rfloor - i} (\chi_2^l)^{-j} & \text{for } d < 2, \\ \sum_{i=1}^{v_l - \lfloor d/2 \rfloor} \sum_{j=1}^{-2\{d/2\} + 2i} (\chi_1^l)^{2\{d/2\} - i} (\chi_2^l)^{-j} + \\ \quad - \sum_{i=0}^{\lfloor d/2 \rfloor - 1} \sum_{j=0}^{2\{d/2\} + 2i} (\chi_1^l)^i (\chi_2^l)^j & \text{for } 2 \leq d < -2v_l, \\ - \sum_{i=1}^{-v_l} \sum_{j=0}^{d-2i} (\chi_1^l)^{\lfloor d/2 \rfloor - i} (\chi_2^l)^j & \text{for } d \geq -2v_l. \end{cases}$$

EXAMPLE 5.18. For $k = 2$ we have just one factor $L_{\alpha\beta}^1$, and two possible cases:

$$L_{\alpha\beta}^1(\chi_1^l, \chi_2^l) = \begin{cases} - \sum_{i=0}^{v_1-1} \sum_{j=0}^{2i+\delta_{1,c}} (\chi_1^1)^i (\chi_2^1)^j & \text{for } v_1 \geq 0, \\ \sum_{i=1}^{-v_1} \sum_{j=1}^{2i-1-\delta_{1,c}} (\chi_1^1)^{-i} (\chi_2^1)^{-j} & \text{for } v_1 < 0. \end{cases}$$

△

EXAMPLE 5.19. For $k = 3$ we start seeing all the possible cases for $L_{\alpha\beta}^1$: for $v_1 \geq 0$ we have

$$L_{\alpha\beta}^1(\chi_1^l, \chi_2^l) = \begin{cases} - \sum_{i=0}^{v_1-1} \sum_{j=0}^{\delta_{1,c}-v_2+2i} (\chi_1^1)^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor + i} (\chi_2^1)^j & \text{for } \delta_{1,c} - v_2 \geq 0, \\ \sum_{i=1}^{-\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - 1} \sum_{j=1}^{-2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^{2\{\frac{\delta_{1,c}-v_2}{2}\} - i} (\chi_2^1)^{-j} + \\ \quad - \sum_{i=0}^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor + v_1 - 1} \sum_{j=0}^{2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^i (\chi_2^1)^j & \text{for } 2 - 2v_1 \leq \delta_{1,c} - v_2 < 0, \\ \sum_{i=0}^{v_l-1} \sum_{j=1}^{-d-2i-1} (\chi_1^1)^{-\lfloor -d/2 \rfloor + i} (\chi_2^1)^{-j} & \text{for } \delta_{1,c} - v_2 < 2 - 2v_1. \end{cases}$$

For $v_1 < 0$ we have similar expressions:

$$L_{\alpha\beta}^1(\chi_1^l, \chi_2^l) = \begin{cases} \sum_{i=1}^{-v_1} \sum_{j=1}^{v_2-\delta_{1,c}+2i-1} (\chi_1^1)^{-\lfloor \frac{v_2-\delta_{1,c}}{2} \rfloor - i} (\chi_2^1)^{-j} & \text{for } \delta_{1,c} - v_2 < 2, \\ \sum_{i=1}^{v_1 - \lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor} \sum_{j=1}^{-2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^{2\{\frac{\delta_{1,c}-v_2}{2}\} - i} (\chi_2^1)^{-j} + \\ \quad - \sum_{i=0}^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - 1} \sum_{j=0}^{2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^i (\chi_2^1)^j & \text{for } 2 \leq \delta_{1,c} - v_2 < -2v_1, \\ - \sum_{i=1}^{-v_1} \sum_{j=0}^{\delta_{1,c}-v_2-2i} (\chi_1^1)^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - i} (\chi_2^1)^j & \text{for } \delta_{1,c} - v_2 \geq -2v_1. \end{cases}$$

$L_{\alpha\beta}^2$ simplifies to

$$L_{\alpha\beta}^2(\chi_1^l, \chi_2^l) = \begin{cases} - \sum_{i=0}^{v_2-1} \sum_{j=0}^{2i+\delta_{2,c}} (\chi_1^2)^i (\chi_2^2)^j & \text{for } v_2 \geq 0, \\ \sum_{i=1}^{-v_2} \sum_{j=1}^{2i-1-\delta_{2,c}} (\chi_1^2)^{-i} (\chi_2^2)^{-j} & \text{for } v_2 < 0. \end{cases}$$

△

A complete proof of the Proposition, together with the explicit computations of the $L_{\alpha\beta}^l$, are given in Appendix D.

COROLLARY 5.20. *The “edge” contribution to the Euler class of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is*

$$\prod_{\alpha,\beta=1}^r \prod_{l=1}^{k-1} \ell_{\alpha\beta}^{(l)}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)})$$

where the expressions of the $\ell_{\alpha\beta}^{(l)}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)})$ are given in equations (68) and (69) below.

With the same notation we used for the equations (66) and (67), we can write explicitly the $\ell^{(l)}$. For aesthetic reasons, we prefer to use here \vec{a} as a variable, instead of the $\vec{a}^{(l)}$ introduced in (65). This is obviously equivalent, but the use of $a_\beta - a_\alpha$ makes the formulae a little nicer. We have for $v_l \geq 0$

$$(68) \quad \ell_{\alpha\beta}^{(l)}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}) = \begin{cases} \prod_{i=0}^{v_l-1} \prod_{j=0}^{d+2i} \left((\lfloor d/2 \rfloor + i)\varepsilon_1^{(l)} + j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right)^{-1} & \text{for } d \geq 0, \\ \prod_{i=1}^{-\lfloor d/2 \rfloor - 1} \prod_{j=1}^{-2\{d/2\}+2i} \left((2\{d/2\} - i)\varepsilon_1^{(l)} - j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right) \cdot \\ \quad \cdot \prod_{i=0}^{\lfloor d/2 \rfloor + v_l - 1} \prod_{j=0}^{2\{d/2\}+2i} \left(i\varepsilon_1^{(l)} + j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right)^{-1} & \text{for } 2 - 2v_l \leq d < 0, \\ \prod_{i=0}^{v_l-1} \prod_{j=1}^{-d-2i-1} \left((-\lfloor -d/2 \rfloor + i)\varepsilon_1^{(l)} - j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right) & \text{for } d < 2 - 2v_l. \end{cases}$$

For $v_l < 0$ we get

$$(69) \quad \ell_{\alpha\beta}^{(l)}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}) = \begin{cases} \prod_{i=1}^{-v_l} \prod_{j=1}^{-d+2i-1} \left((-\lfloor -d/2 \rfloor - i)\varepsilon_1^{(l)} - j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right) & \text{for } d < 2, \\ \prod_{i=1}^{v_l - \lfloor d/2 \rfloor} \prod_{j=1}^{-2\{d/2\}+2i} \left((2\{d/2\} - i)\varepsilon_1^{(l)} - j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right) \cdot \\ \quad \cdot \prod_{i=0}^{\lfloor d/2 \rfloor - 1} \prod_{j=0}^{2\{d/2\}+2i} \left(i\varepsilon_1^{(l)} + j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right)^{-1} & \text{for } 2 \leq d < -2v_l, \\ \prod_{i=1}^{-v_l} \prod_{j=0}^{d-2i} \left((\lfloor d/2 \rfloor - i)\varepsilon_1^{(l)} + j\varepsilon_2^{(l)} + a_\beta - a_\alpha \right)^{-1} & \text{for } d \geq -2v_l. \end{cases}$$

EXAMPLE 5.21. For $k = 2$ we have just $\ell^{(1)}$. Introducing $a_{\beta\alpha}^{(1)} := a_\beta - a_\alpha - \delta_{1,c}\varepsilon_1^{(1)}$ as in [15], we obtain

$$\ell_{\alpha\beta}^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}^{(1)}) = \begin{cases} \prod_{i=\delta_{1,c}}^{v_1+\delta_{1,c}-1} \prod_{j=0}^{2i+\delta_{1,c}} \left(i\varepsilon_1^{(1)} + j\varepsilon_2^{(1)} + a_{\beta\alpha}^{(1)} \right)^{-1} & \text{for } v_1 \geq 0, \\ \prod_{i=1-\delta_{1,c}}^{-v_1-\delta_{1,c}} \prod_{j=1}^{2i-1-\delta_{1,c}} \left(-i\varepsilon_1^{(1)} - j\varepsilon_2^{(1)} + a_{\beta\alpha}^{(1)} \right) & \text{for } v_1 < 0. \end{cases}$$

△

EXAMPLE 5.22. For $k = 3$ we have $\ell^{(1)}$ and $\ell^{(2)}$. For the first, with $v_l \geq 0$

$$\ell_{\alpha\beta}^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \vec{a}) = \begin{cases} \prod_{i=0}^{v_1-1} \prod_{j=0}^{\delta_{1,c}-v_2+2i} \left(\left(\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor + i \right) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right)^{-1} & \text{for } \delta_{1,c} - v_2 \geq 0, \\ \prod_{i=1}^{-\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor-1} \prod_{j=1}^{-2\left\{ \frac{\delta_{1,c}-v_2}{2} \right\}+2i} \left(\left(2\left\{ \frac{\delta_{1,c}-v_2}{2} \right\} - i \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right) \cdot \\ \cdot \prod_{i=0}^{\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor+v_1-1} \prod_{j=0}^{2\left\{ \frac{\delta_{1,c}-v_2}{2} \right\}+2i} \left(i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right)^{-1} & \text{for } 2 - 2v_1 \leq \delta_{1,c} - v_2 < 0, \\ \prod_{i=0}^{v_1-1} \prod_{j=1}^{-d-2i-1} \left((-\lfloor -d/2 \rfloor + i) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right) & \text{for } \delta_{1,c} - v_2 < 2 - 2v_1. \end{cases}$$

For $v_1 < 0$ we have

$$\ell_{\alpha\beta}^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \vec{a}) = \begin{cases} \prod_{i=1}^{-v_1} \prod_{j=1}^{v_2-\delta_{1,c}+2i-1} \left(\left(-\left\lfloor \frac{v_2-\delta_{1,c}}{2} \right\rfloor - i \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right) & \text{for } \delta_{1,c} - v_2 < 2, \\ \prod_{i=1}^{v_1-\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor} \prod_{j=1}^{-2\left\{ \frac{\delta_{1,c}-v_2}{2} \right\}+2i} \left(\left(2\left\{ \frac{\delta_{1,c}-v_2}{2} \right\} - i \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right) \cdot \\ \cdot \prod_{i=0}^{\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor-1} \prod_{j=0}^{2\left\{ \frac{\delta_{1,c}-v_2}{2} \right\}+2i} \left(i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right)^{-1} & \text{for } 2 \leq \delta_{1,c} - v_2 < -2v_1, \\ \prod_{i=1}^{-v_1} \prod_{j=0}^{\delta_{1,c}-v_2-2i} \left(\left(\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor - i \right) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} + a_\beta - a_\alpha \right)^{-1} & \text{for } \delta_{1,c} - v_2 \geq -2v_1. \end{cases}$$

$\ell^{(2)}$ simplifies to

$$\ell_{\alpha\beta}^{(2)}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \vec{a}) = \begin{cases} \prod_{i=0}^{v_2-1} \prod_{j=0}^{2i+\delta_{2,c}} (i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} + a_\beta - a_\alpha)^{-1} & \text{for } v_2 \geq 0, \\ \prod_{i=1}^{-v_2} \prod_{j=1}^{2i-1-\delta_{2,c}} (-i \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} + a_\beta - a_\alpha) & \text{for } v_2 < 0. \end{cases}$$

△

Again, the proof of the Corollary and the computations of the $L_{\alpha\beta}^{(l)}$ are given in Appendix D.

REMARK 5.23. We want to stress here that the condition in the edge contribution depend on the coefficients of the Cartan matrix. In [15], basing on a conjectural splitting of the full partition function on X_k as a product of full partition functions on the open affine subsets U_i , the authors obtain an expression for the edge factors which depends just on the fan. At this stage it seems not easy to us to compare the two results in general, due to the different structures of the expressions. We can just say, as explained in the Introduction, that for $k = 2$ they agree. △

5.3. $\mathcal{N} = 2$ pure gauge theories

In this very computational section we introduce the deformed partition function for supersymmetric gauge theories on \mathcal{X}_k , compute it, and give examples for $k = 2, 3$. Then we focus on the *instanton part* of the deformed partition function. We compute it, obtaining a factorization formula that involves the instanton part of the Nekrasov partition functions on the open affine subsets U_i , weighted by same edge factors that appears in the formula for the

Euler class of the tangent to $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$. Then we consider the *deformed* instanton part, for which we obtain another factorization formula that involves both the classical and instanton parts of the Nekrasov partition functions on the U_i 's, again weighted by the edge factors. As a by-product, we obtain a mathematically rigorous way, by using framed sheaves, for deriving the classical and instanton partition functions of the gauge theories we are dealing with. In conclusion, we give expressions for the partition functions for pure $U(r)$ -gauge theories on X_k . All the computations are done on the moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$, where we set the degree of the framing sheaf to zero.

Let $\vec{v} \in \frac{1}{k}\mathbb{Z}^{\oplus k-1}$. As we saw in Remark 5.3, we can define an element of the root lattice \mathfrak{Q} , which is $\gamma := C^{-1}\vec{e}_c - \vec{v}$ if $c > 0$, $\gamma = -\vec{v}$ otherwise, where c is the equivalence class modulo k of kv_{k-1} . Viceversa, from $\gamma \in \mathfrak{Q}$ and $c \in \{0, 1, \dots, k-1\}$ we can define $\vec{v} \in \frac{1}{k}\mathbb{Z}^{\oplus k-1}$. By fixing the framing sheaf $\mathcal{F}_\infty^{0,\vec{w}}$, we get that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$ and, equivalently, c is the equivalence class modulo k of $\sum_{i=0}^{k-1} iw_i$. Through this and the next two sections, we always keep in mind the bijective correspondence between $\vec{v} \in \frac{1}{k}\mathbb{Z}^{\oplus k-1}$ and $(c, \gamma) \in \{0, \dots, k-1\} \times \mathfrak{Q}$. In particular, for any expression of the partition functions we will give, depending on \vec{v} , one can give an equivalent version depending on $\gamma \in \mathfrak{Q}$ and $c \in \{0, 1, \dots, k-1\}$. We choose the dependence on \vec{v} for aesthetic reasons: the formulae are nicer.

5.3.1. Definition of the partition function. With the same notations as in the previous Section, let $[(\mathcal{E}, \phi_{\mathcal{E}})]$ be a T -fixed point of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ and (\vec{Y}, \vec{v}) its corresponding combinatorial data. By Corollaries 5.16 and 5.20, we have

$$\text{Euler}(T_{(\vec{Y}, \vec{v})} \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})) = \prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}) .$$

Let $\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1}$ such that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$. Define

$$(70) \quad \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) := \sum_{\Delta \in \frac{1}{2rk}\mathbb{Z}} \mathbf{q}^{\Delta + \frac{1}{2r} \vec{v} \cdot C \vec{v}} \cdot \int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})} \exp \left(\sum_{p=0}^{\infty} \left(\sum_{i=1}^{k-1} t_p^{(i)} \left[\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i] \right]_p + \tau_p \left[\text{ch}_T(\tilde{\mathcal{E}})/[X_k] \right]_{p-1} \right) \right) ,$$

where $\tilde{\mathcal{E}}$ is the *universal sheaf*, $\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i]$ denotes the *slant product* $/$ between $\text{ch}_T(\tilde{\mathcal{E}})$ and $[\mathcal{D}_i]$ and the class $\text{ch}_T(\tilde{\mathcal{E}})/[X_k]$ is defined formally by localization as

$$\text{ch}_T(\tilde{\mathcal{E}})/[X_k] := \sum_{i=1}^k \frac{1}{\text{Euler}(T_{p_i} X_k)} i_{\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})}^* \text{ch}_T(\tilde{\mathcal{E}}) ;$$

here $i_{\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})}$ denotes the inclusion map of $\{p_i\} \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ in $X_k \times \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$.

REMARK 5.24. Let \mathcal{X} be a topological stack with an action of an ordinary torus T . As explained in [48, Section 5], there is a well-posed notion of T -equivariant (co)homology theory on \mathcal{X} . When \mathcal{X} is a topological space, their definition reduces to Borel's definition of

T -equivariant (co)homology theory on topological spaces. So the slant product is well defined also for T -equivariant (co)homology theories on topological stacks. \triangle

By the localization formula we get

$$\begin{aligned} \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) = \\ \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \cdot \\ \cdot \iota_{(\vec{Y}, \vec{v})}^* \exp \left(\sum_{p=0}^{\infty} \left(\sum_{i=1}^{k-1} t_p^{(i)} \left[\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i] \right]_p + \tau_p \left[\text{ch}_T(\tilde{\mathcal{E}})/[X_k] \right]_{p-1} \right) \right). \end{aligned}$$

Computation of $\iota_{(\vec{Y}, \vec{v})}^ \text{ch}_T(\tilde{\mathcal{E}})/[X_k]$.* First note that

$$\iota_{(\vec{Y}, \vec{v})}^* \text{ch}_T(\tilde{\mathcal{E}})/[X_k] = \sum_{i=1}^k \frac{1}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}} \iota_{\{p_i\} \times \{(\vec{Y}, \vec{v})\}}^* \text{ch}_T(\tilde{\mathcal{E}}).$$

Let $i \in \{1, \dots, k\}$, $[(\mathcal{E}, \phi_{\mathcal{E}})] = [\oplus_{\alpha=1}^r (i_*(\mathcal{I}_{\alpha}) \otimes \mathcal{R}^{C\vec{v}_{\alpha}}, \phi_{\alpha})]$ a T -fixed point and (\vec{Y}, \vec{v}) its corresponding combinatorial data. Then

$$\iota_{\{p_i\} \times \{(\vec{Y}, \vec{v})\}}^* \text{ch}_T(\tilde{\mathcal{E}}) = \sum_{\alpha=1}^r e_{\alpha} \iota_{p_i}^* \text{ch}_{T_i}(i_*(\mathcal{I}_{\alpha}) \otimes \mathcal{R}^{C\vec{v}_{\alpha}}) = \sum_{\alpha=1}^r e_{\alpha} \text{ch}_{T_i}((\mathcal{I}_{\alpha})_{p_i}) \text{ch}_{T_i}(\mathcal{R}_{p_i}^{C\vec{v}_{\alpha}}),$$

where ι_{p_i} denotes the inclusion morphism of the point p_i into X_k .

By [86, Formula 4.1] we get

$$(71) \quad \text{ch}_{T_i}((\mathcal{I}_{\alpha})_{p_i}) = 1 - (1 - (\chi_1^i)^{-1})(1 - (\chi_2^i)^{-1}) \sum_{t \in Y_{\alpha}^i} (\chi_1^i)^{-\ell'(t)} (\chi_2^i)^{-a'(t)}.$$

By Lemma 4.12 and Formula (45), we have $\text{ch}_{T_i}(\mathcal{R}_{p_i}^{C\vec{v}_{\alpha}}) = (\chi_1^i)^{-(\vec{v}_{\alpha})_i} (\chi_2^i)^{-(\vec{v}_{\alpha})_{i-1}}$. Summing up, we get

$$\iota_{(\vec{Y}, \vec{v})}^* \text{ch}_T(\tilde{\mathcal{E}})/[X_k] = \sum_{i=1}^k \sum_{\alpha=1}^r \frac{e_{\alpha}^{a_{\alpha}^{(i)}}}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}} \left(1 - (1 - e^{-\varepsilon_1^{(i)}})(1 - e^{-\varepsilon_2^{(i)}}) \sum_{t \in Y_{\alpha}^i} e^{-\varepsilon_1^{(i)} \ell'(t) - \varepsilon_2^{(i)} a'(t)} \right).$$

Let us introduce the following notation:

$$\text{ch}_{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}) := \sum_{\alpha=1}^r \frac{e_{\alpha}^{a_{\alpha}^{(i)}}}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}} \left(1 - (1 - e^{-\varepsilon_1^{(i)}})(1 - e^{-\varepsilon_2^{(i)}}) \sum_{t \in Y_{\alpha}^i} e^{-\varepsilon_1^{(i)} \ell'(t) - \varepsilon_2^{(i)} a'(t)} \right).$$

Then

$$(72) \quad \iota_{(\vec{Y}, \vec{v})}^* \text{ch}_T(\tilde{\mathcal{E}})/[X_k] = \sum_{i=1}^k \text{ch}_{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}).$$

REMARK 5.25. By Formula (71) we have

$$(73) \quad \begin{aligned} \frac{1}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}} \sum_{\alpha=1}^r a_{\alpha}^{(i)} &= [\text{ch}_{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})]_{-1} = 0, \\ \frac{1}{2\varepsilon_1^{(i)} \varepsilon_2^{(i)}} \sum_{\alpha=1}^r (a_{\alpha}^{(i)})^2 - \sum_{\alpha=1}^r |Y_{\alpha}^i| &= [\text{ch}_{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})]_0. \end{aligned}$$

These formulas will be useful later on. \triangle

Computation of $\iota_{(\vec{Y}, \vec{v})}^ \text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i]$.* Let $[(\mathcal{E}, \phi_{\mathcal{E}})] = [\oplus_{\alpha=1}^r (i_*(\mathcal{I}_{\alpha}) \otimes \mathcal{R}^{C\vec{v}_{\alpha}}, \phi_{\alpha})]$ be a T -fixed point and (\vec{Y}, \vec{v}) its corresponding combinatorial data. Then

$$\begin{aligned} \text{ch}_T(\mathcal{E}) &= \sum_{\alpha=1}^r e_{\alpha} \text{ch}_{T_t}(\mathcal{R}^{C\vec{v}_{\alpha}}) \text{ch}_{T_t}(i_*(\mathcal{I}_{\alpha})) \\ &= \sum_{\alpha=1}^r e_{\alpha} e^{-\sum_{j=1}^{k-1} (\vec{v}_{\alpha})_j [\mathcal{D}_j]} \left(1 - \sum_{l=1}^k [p_l] (1 - (\chi_1^l)^{-1}) (1 - (\chi_2^l)^{-1}) \sum_{t \in Y_{\alpha}^l} (\chi_1^l)^{-\ell'(t)} (\chi_2^l)^{-a'(t)} \right). \end{aligned}$$

In the following we compute separately $e^{-\sum_{j=1}^{k-1} (\vec{v}_{\alpha})_j [\mathcal{D}_j]}/[\mathcal{D}_i]$ and $e^{-\sum_{j=1}^{k-1} (\vec{v}_{\alpha})_j [\mathcal{D}_j]} [p_l]/[\mathcal{D}_i]$.

$$\begin{aligned} e^{-\sum_{j=1}^{k-1} (\vec{v}_{\alpha})_j [\mathcal{D}_j]}/[\mathcal{D}_i] &= \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \left(\sum_{j=1}^{k-1} (\vec{v}_{\alpha})_j [\mathcal{D}_j] \right)^m / [\mathcal{D}_i] = \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \int_{X_k} \left(\sum_{j=1}^{k-1} (\vec{v}_{\alpha})_j [D_j] \right)^m \cdot [D_i] = \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \sum_{m_1+m_2+\dots+m_{k-1}=m} a_{m_1, m_2, \dots, m_{k-1}} (\vec{v}_{\alpha})_1^{m_1} \dots (\vec{v}_{\alpha})_{k-1}^{m_{k-1}} \\ &\quad \cdot \int_{X_k} [D_1]^{m_1} \dots [D_i]^{m_i+1} \dots [D_{k-1}]^{m_{k-1}}. \end{aligned}$$

Since

$$(74) \quad \iota_{p_i}^* [D_l] = \begin{cases} \varepsilon_1^{(l)} & \text{if } i = l, \\ \varepsilon_2^{(l+1)} & \text{if } i = l+1, \\ 0 & \text{otherwise,} \end{cases}$$

the previous integral is nonzero if there exists an index $n \in \{1, \dots, k-1\}$ such that only the exponent of $[D_n]$ is nonzero or there exists an index $n' \in \{2, \dots, k-1\}$ such that only the

exponents of $[D_{n'-1}]$ and $[D_{n'}]$ are nonzero. Therefore we obtain

$$\begin{aligned}
& e^{-\sum_{j=1}^{k-1} (\vec{v}_\alpha)_j [\mathcal{D}_j]} / [\mathcal{D}_i] = \\
& = \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \left(\sum_{n=1}^{k-1} \frac{(\vec{v}_\alpha)_n^m (\varepsilon_1^{(n)})^m (\varepsilon_1^{(n)})^{\delta_{n,i}}}{\varepsilon_1^{(n)} \varepsilon_2^{(n)}} + \sum_{n=1}^{k-1} \frac{(\vec{v}_\alpha)_n^m (\varepsilon_2^{(n+1)})^m (\varepsilon_2^{(n+1)})^{\delta_{n,i}}}{\varepsilon_1^{(n+1)} \varepsilon_2^{(n+1)}} + \right. \\
& \quad \left. + \sum_{n=2}^{k-1} \sum_{l=0}^m \binom{m}{l} \frac{(\vec{v}_\alpha)_n^l (\vec{v}_\alpha)_{n-1}^{m-l} (\varepsilon_1^{(n)})^l (\varepsilon_1^{(n)})^{\delta_{n,i}} (\varepsilon_2^{(n)})^{m-l} (\varepsilon_2^{(n)})^{\delta_{n-1,i}}}{\varepsilon_1^{(n)} \varepsilon_2^{(n)}} \right) = \\
& = \sum_{n=1}^{k-1} \frac{(\varepsilon_1^{(n)})^{\delta_{n,i}}}{\varepsilon_1^{(n)} \varepsilon_2^{(n)}} e^{-(\vec{v}_\alpha)_n \varepsilon_1^{(n)}} + \sum_{n=2}^k \frac{(\varepsilon_2^{(n)})^{\delta_{n-1,i}}}{\varepsilon_1^{(n)} \varepsilon_2^{(n)}} e^{-(\vec{v}_\alpha)_{n-1} \varepsilon_2^{(n)}} + \\
& \quad + \sum_{n=2}^{k-1} \frac{(\varepsilon_1^{(n)})^{\delta_{n,i}} (\varepsilon_2^{(n)})^{\delta_{n-1,i}}}{\varepsilon_1^{(n)} \varepsilon_2^{(n)}} e^{-((\vec{v}_\alpha)_n \varepsilon_1^{(n)} + (\vec{v}_\alpha)_{n-1} \varepsilon_2^{(n)})} .
\end{aligned}$$

On the other hand, by using the same arguments as before for $l \in \{2, \dots, k-2\}$ we have

$$\begin{aligned}
e^{-\sum_{j=1}^{k-1} (\vec{v}_\alpha)_j [\mathcal{D}_j]} [p_l] / [\mathcal{D}_i] &= \frac{(\varepsilon_1^{(l)})^{\delta_{l,i}}}{\varepsilon_1^{(l)} \varepsilon_2^{(l)}} e^{-(\vec{v}_\alpha)_l \varepsilon_1^{(l)}} + \frac{(\varepsilon_2^{(l)})^{\delta_{l-1,i}}}{\varepsilon_1^{(l)} \varepsilon_2^{(l)}} e^{-(\vec{v}_\alpha)_{l-1} \varepsilon_2^{(l)}} \\
&\quad + \frac{(\varepsilon_1^{(l)})^{\delta_{l,i}} (\varepsilon_2^{(l)})^{\delta_{l-1,i}}}{\varepsilon_1^{(l)} \varepsilon_2^{(l)}} e^{-((\vec{v}_\alpha)_l \varepsilon_1^{(l)} + (\vec{v}_\alpha)_{l-1} \varepsilon_2^{(l)})} \text{ for } l \in \{2, \dots, k-2\} , \\
e^{-\sum_{j=1}^{k-1} (\vec{v}_\alpha)_j [\mathcal{D}_j]} [p_1] / [\mathcal{D}_i] &= \frac{(\varepsilon_1^{(1)})^{\delta_{1,i}}}{\varepsilon_1^{(1)} \varepsilon_2^{(1)}} e^{-(\vec{v}_\alpha)_1 \varepsilon_1^{(1)}} , \\
e^{-\sum_{j=1}^{k-1} (\vec{v}_\alpha)_j [\mathcal{D}_j]} [p_k] / [\mathcal{D}_i] &= \frac{(\varepsilon_2^{(k)})^{\delta_{k-1,i}}}{\varepsilon_1^{(k)} \varepsilon_2^{(k)}} e^{-(\vec{v}_\alpha)_{k-1} \varepsilon_2^{(k)}} .
\end{aligned}$$

Explicit formula. Let $\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1}$ such that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$. By using the previous computations we get

$$\begin{aligned}
& \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) = \\
&= \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \cdot \\
& \quad \cdot \prod_{l=1}^k \exp \left(\sum_{p=0}^{\infty} \left((t_p^{(l)} \varepsilon_1^{(l)} + t_p^{(l-1)} \varepsilon_2^{(l)} + \tau_p) \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}) \right]_{p-1} \right) \right) \cdot \\
& \quad \cdot \exp \left(\sum_{p=0}^{\infty} \left(\sum_{l=1}^k \left(\sum_{i=1}^{l-2} t_p^{(i)} + \sum_{i=l+1}^{k-1} t_p^{(i)} \right) \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}) \right]_p + \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^{k-1} t_p^{(i)} \sum_{l=2}^{k-1} \left[\left(\varepsilon_1^{(l)} \right)^{\delta_{l,i}} \text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} - (\vec{v})_{l-1} \varepsilon_2^{(l)}) \right. \right. \right. \\
& \quad \left. \left. \left. + \left(\varepsilon_2^{(l)} \right)^{\delta_{l-1,i}} \text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} - (\vec{v})_l \varepsilon_1^{(l)}) \right] \right]_p \right) \right) \right),
\end{aligned}$$

where we set $t_p^{(0)} = t_p^{(k)} = 0$ for any p .

EXAMPLE 5.26. For $k = 2$ we get

$$\begin{aligned}
& \mathcal{Z}_v(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}) = \\
&= \sum_{\mathbf{v}} \frac{\mathbf{q}^{\sum_{\alpha=1}^r v_{\alpha}^2}}{\prod_{\alpha\beta} \ell_{\alpha\beta}^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \vec{a}^{(1)})} \cdot \\
& \quad \cdot \mathcal{Z}_{\mathbb{R}^4}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \vec{a}^{(1)}; \mathbf{q}, \vec{\tau} + \varepsilon_1^{(1)} \vec{t}^{(1)}) \cdot \mathcal{Z}_{\mathbb{R}^4}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \vec{a}^{(2)}; \mathbf{q}, \vec{\tau} + \varepsilon_2^{(2)} \vec{t}^{(1)}),
\end{aligned}$$

where $\mathcal{Z}_{\mathbb{R}^4}$ is the deformed the Nekrasov partition function for \mathbb{R}^4 defined in the Introduction, and by [86, Section 4.2]

$$\mathcal{Z}_{\mathbb{R}^4}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \tau) := \sum_{\vec{Y}} \frac{\mathbf{q}^{\sum_{\alpha=1}^r |Y_{\alpha}|}}{\prod_{\alpha\beta} m_{\alpha\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a})} \exp \left(\sum_{p=0}^{\infty} \tau_p \left[\text{ch}_{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a}) \right]_{p-1} \right).$$

△

EXAMPLE 5.27. For $k = 3$ we get

$$\begin{aligned}
& \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \vec{t}^{(2)}) = \\
& = \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^2 \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^3 m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \cdot \\
& \quad \cdot \prod_{l=1}^3 \exp \left(\sum_{p=0}^{\infty} \left((t_p^{(l)} \varepsilon_1^{(l)} + t_p^{(l-1)} \varepsilon_2^{(l)} + \tau_p) \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}) \right]_{p-1} \right) \right) \cdot \\
& \quad \cdot \exp \left(\sum_{p=0}^{\infty} \left(t_p^{(2)} \left[\text{ch}_{\vec{Y}^1}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \vec{a}^{(1)}) \right]_p + t_p^{(1)} \left[\text{ch}_{\vec{Y}^2}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \vec{a}^{(2)}) \right]_p + \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^2 t_p^{(i)} \left[\left(\left(\varepsilon_1^{(2)} \right)^{\delta_{2,i}} \text{ch}_{\vec{Y}^2}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \vec{a}^{(2)} - (\vec{v})_1 \varepsilon_2^{(2)}) + \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \left(\varepsilon_2^{(2)} \right)^{\delta_{1,i}} \text{ch}_{\vec{Y}^2}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \vec{a}^{(2)} - (\vec{v})_2 \varepsilon_1^{(2)}) \right) \right]_p \right) \right] \right) \right) \cdot
\end{aligned}$$

△

5.3.2. Instanton part. Let $\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1}$ such that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$. The *instanton part* of $\mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)})$ is defined as

$$\begin{aligned}
(75) \quad \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) &:= \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, 0, \dots, 0) = \\
&= \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} ,
\end{aligned}$$

where we choose $\vec{\tau} = 0$ and $\vec{t}^{(i)} = 0$ for $i = 1, \dots, k-1$. Since the *instanton part of the Nekrasov partition function* (see the Introduction) for pure $SU(r)$ -gauge theories on \mathbb{R}^4 is, by [24, Formula (3.16)]

$$\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) := \sum_{\vec{Y}} \frac{\mathbf{q}^{\sum_{\alpha=1}^r |Y_{\alpha}|}}{m_{\alpha\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a})} ,$$

we get

$$(76) \quad \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) = \sum_{\vec{v}} \frac{\mathbf{q}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)})} \prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}; \mathbf{q}) .$$

5.3.3. The deformed instanton part. Let $\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1}$ be such that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$. The *deformed instanton part* of $\mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)})$ is defined as

$$(77) \quad \mathcal{Z}_{\vec{v}}^{def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \tau_1) := \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, 0, \dots, 0) \\ = \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \\ \cdot \prod_{l=1}^k \exp\left(-\tau_1 \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)})\right]_0\right),$$

where we choose $\vec{\tau} = (0, -\tau_1, 0, \dots)$ and $\vec{t}^{(i)} = 0$ for $i = 1, \dots, k-1$.

By Formula (73) we get

$$(78) \quad \mathcal{Z}_{\vec{v}}^{def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \tau_1) = \\ \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \\ \cdot \prod_{l=1}^k \exp\left(-\tau_1 \left(\frac{1}{2\varepsilon_1^{(l)}\varepsilon_2^{(l)}} \sum_{\alpha=1}^r (a_{\alpha}^{(l)})^2 - \sum_{\alpha=1}^r |Y_{\alpha}^l|\right)\right) \\ = \sum_{\vec{v}} \frac{\mathbf{q}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)})} \prod_{l=1}^k \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}; \tau_1) \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}; \mathbf{q}_{eff}),$$

where $\mathbf{q}_{eff} := \mathbf{q}e^{\tau_1}$. Here we used the *classical part of the Nekrasov partition function* for pure $SU(r)$ -gauge theories on \mathbb{R}^4 , which is (cf. [15, Formula (3.1)]):

$$(79) \quad \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1) := \exp\left(-\frac{\tau_1}{2\varepsilon_1\varepsilon_2} \sum_{\alpha=1}^r a_{\alpha}^2\right).$$

It is possible to give another expression of $\prod_{l=1}^k \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}; \tau_1)$. From the identities

$$(80) \quad \sum_{l=1}^k \frac{1}{\varepsilon_1^{(l)}\varepsilon_2^{(l)}} = \frac{1}{k\varepsilon_1\varepsilon_2} \quad \text{and} \quad \sum_{l=1}^k \frac{(\vec{v}_{\alpha})_l \varepsilon_1^{(l)} + (\vec{v}_{\alpha})_{l-1} \varepsilon_2^{(l)}}{\varepsilon_1^{(l)}\varepsilon_2^{(l)}} = 0,$$

it follows that

$$(81) \quad \sum_{l=1}^k \frac{1}{2\varepsilon_1^{(l)}\varepsilon_2^{(l)}} \sum_{\alpha=1}^r (a_{\alpha}^{(i)})^2 = \\ \sum_{\alpha=1}^r \frac{a_{\alpha}^2}{2k\varepsilon_1\varepsilon_2} + \sum_{\alpha, \beta=1}^r \sum_{l=1}^k \frac{\left((\vec{v}_{\alpha})_l \varepsilon_1^{(l)} + (\vec{v}_{\alpha})_{l-1} \varepsilon_2^{(l)}\right) \left((\vec{v}_{\beta})_l \varepsilon_1^{(l)} + (\vec{v}_{\beta})_{l-1} \varepsilon_2^{(l)}\right)}{2\varepsilon_1^{(l)}\varepsilon_2^{(l)}}.$$

By localization formula (cf. Formula (74)) we get

$$(82) \quad \sum_{l=1}^k \frac{\left((\vec{v}_\alpha)_l \varepsilon_1^{(l)} + (\vec{v}_\alpha)_{l-1} \varepsilon_2^{(l)} \right) \left((\vec{v}_\beta)_l \varepsilon_1^{(l)} + (\vec{v}_\beta)_{l-1} \varepsilon_2^{(l)} \right)}{\varepsilon_1^{(l)} \varepsilon_2^{(l)}} = \left(\sum_{i=1}^{k-1} (\vec{v}_\alpha)_i [D_i] \right) \left(\sum_{i=1}^{k-1} (\vec{v}_\beta)_i [D_i] \right) = -\vec{v}_\alpha \cdot C \vec{v}_\beta .$$

Thus

$$(83) \quad \prod_{l=1}^k \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}; \tau_1) = \exp \left(- \sum_{l=1}^k \frac{\tau_1}{2 \varepsilon_1^{(l)} \varepsilon_2^{(l)}} \sum_{\alpha=1}^r (a_\alpha^{(l)})^2 \right) \\ = \exp \left(- \frac{\tau_1}{2 k \varepsilon_1 \varepsilon_2} \sum_{\alpha=1}^r a_\alpha^2 \right) \exp \left(\frac{\tau_1}{2} \sum_{\alpha, \beta=1}^r \vec{v}_\alpha \cdot C \vec{v}_\beta \right) = \\ \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1)^{\frac{1}{k}} \exp \left(\frac{\tau_1}{2} \sum_{\alpha, \beta=1}^r \vec{v}_\alpha \cdot C \vec{v}_\beta \right) ,$$

and therefore

$$(84) \quad \mathcal{Z}_{\vec{v}}^{def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \tau_1) = \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1)^{\frac{1}{k}} \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}_{eff}, \tau_1) ,$$

where

$$(85) \quad \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}_{eff}, \tau_1) = \\ = \sum_{\vec{v}} \frac{(e\tau_1)^{\frac{1}{2} \sum_{\alpha \neq \beta} \vec{v}_\alpha \cdot C \vec{v}_\beta} \mathbf{q}_{eff}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C \vec{v}_\alpha}}{\prod_{\alpha \beta} \prod_{j=1}^{k-1} \ell_{\alpha \beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)})} \prod_{l=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2, inst}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}; \mathbf{q}_{eff}) ,$$

Note that for $\tau_1 = 0$, the previous partition function coincides with (76).

REMARK 5.28. As we see in the previous formula, the deformed instanton part, which is a well-defined partition function defined using moduli spaces of framed sheaves, includes the “classical” and the “instanton” contributions of the gauge theories. In particular, we obtain a way to define and compute the classical contribution by using moduli spaces of framed sheaves. \triangle

5.3.4. Pure gauge theory. Let us define the *deformed partition function* for pure $U(r)$ -gauge theories on X_k by

$$(86) \quad \mathcal{Z}_{ALE}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\xi}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) := \\ = \sum_{\substack{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} i w_i \pmod k}} \vec{\xi}^{-\vec{v}} \mathcal{Z}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) ,$$

where we denoted $\vec{\xi}^{-\vec{v}} := \xi_1^{-v_1} \cdots \xi_{k-1}^{-v_{k-1}}$. Define also its *instanton part* by

$$(87) \quad \mathcal{Z}_{ALE}^{\mathcal{N}=2, inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\xi}) := \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}}} \vec{\xi}^{-\vec{v}} \mathcal{Z}_{\vec{v}}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) ,$$

and its *deformed instanton part* by

$$(88) \quad \mathcal{Z}_{ALE}^{\mathcal{N}=2, def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\xi}) := \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}}} \vec{\xi}^{-\vec{v}} \mathcal{Z}_{\vec{v}}^{def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) \\ = \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a})^{\frac{1}{k}} \mathcal{Z}_{ALE}^{\mathcal{N}=2, inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}_{eff}, \vec{\xi}) .$$

5.4. $\mathcal{N} = 2$ gauge theories with one adjoint hypermultiplet

Here we follow the computations of the partition functions in the previous sections, adding an adjoint mass. We compute and obtain factorization formulae similar to the previous, for the deformed partition function and its instanton part. We close the section giving expression for the $\mathcal{N} = 2^*$ $U(r)$ -gauge theories on X_k .

5.4.1. The partition function $\mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)})$. By following [47, Section 4.5], let $T_m = \mathbb{C}^*$ be an algebraic torus, then $H_{T_m}^*(\text{pt}; \mathbb{Q}) = \mathbb{Q}[m]$. For a T -equivariant locally free sheaf \mathcal{E} of rank n on $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})$ we define the class

$$E_m(\mathcal{E}) := m^n + (c_1)_T(\mathcal{E})m^{n-1} + \cdots + (c_n)_T(\mathcal{E}) \in H_{T \times T_m}^*(\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})) .$$

Let $\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1}$ such that $kv_{k-1} \equiv \sum_{i=0}^{k-1} iw_i \pmod{k}$. Define

$$(89) \quad \mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) := \\ = \sum_{\Delta \in \frac{1}{2rk}\mathbb{Z}} \mathbf{q}^{\Delta + \frac{1}{2r} \vec{v} \cdot C \vec{v}} \int_{\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})} E_m(\mathcal{T}\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})) \cdot \\ \cdot \exp \left(\sum_{p=0}^{\infty} \left(\sum_{i=1}^{k-1} t_p^{(i)} \left[\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i] \right]_p + \tau_p \left[\text{ch}_T(\tilde{\mathcal{E}})/[X_k] \right]_{p-1} \right) \right) ,$$

where $\mathcal{T}\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})$ is the tangent bundle of $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})$.

By localization formula we get

$$\mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) = \\ = \sum_{(\vec{Y}, \vec{v})} \frac{\mathbf{q}^{\sum_{\alpha=1}^r n_\alpha + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C \vec{v}_\alpha}}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}_i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \cdot \\ \cdot i_{(\vec{Y}, \vec{v})}^* E_m(\mathcal{T}\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0, \vec{w}})) \cdot \\ \cdot i_{(\vec{Y}, \vec{v})}^* \exp \left(\sum_{p=0}^{\infty} \left(\sum_{i=1}^{k-1} t_p^{(i)} \left[\text{ch}_T(\tilde{\mathcal{E}})/[\mathcal{D}_i] \right]_p + \tau_p \left[\text{ch}_T(\tilde{\mathcal{E}})/[X_k] \right]_{p-1} \right) \right) .$$

Let us denote by d the dimension of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$. Note that

$$i_{(\vec{Y},\vec{v})}^* E_m(\mathcal{T}\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})) = \sum_{l=0}^d m^{d-l} (c_l)_T(\mathcal{T}_{(\vec{Y},\vec{v})} \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})) .$$

Since $\mathcal{T}_{(\vec{Y},\vec{v})} \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ as a T -module is a direct sum of one-dimensional T -modules (see Section 5.2.2), we get

$$i_{(\vec{Y},\vec{v})}^* E_m(\mathcal{T}\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})) = \prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)} + m) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)} + m) .$$

Thus

$$\begin{aligned} \mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) = \\ \sum_{(\vec{Y},\vec{v})} \mathbf{q}^{\sum_{\alpha=1}^r n_\alpha + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C \vec{v}_\alpha} \frac{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)} + m) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)} + m)}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \\ \cdot \prod_{l=1}^k \exp \left(\sum_{p=0}^{\infty} \left((t_p^{(l)} \varepsilon_1^{(l)} + t_p^{(l-1)} \varepsilon_2^{(l)} + \tau_p) \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}) \right]_{p-1} \right) \right) \\ \cdot \exp \left(\sum_{p=0}^{\infty} \left(\sum_{l=1}^k \left(\sum_{i=1}^{l-2} t_p^{(i)} + \sum_{i=l+1}^{k-1} t_p^{(i)} \right) \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}) \right]_p + \right. \right. \\ \left. \left. + \sum_{i=1}^{k-1} t_p^{(i)} \sum_{l=2}^{k-1} \left[\left(\left(\varepsilon_1^{(l)} \right)^{\delta_{l,i}} \text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} - (\vec{v}_\alpha)_{l-1} \varepsilon_2^{(l)}) + \right. \right. \right. \right. \\ \left. \left. \left. + \left(\varepsilon_2^{(l)} \right)^{\delta_{l-1,i}} \text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} - (\vec{v}_\alpha)_l \varepsilon_1^{(l)}) \right) \right]_p \right) \right) \right) , \end{aligned}$$

its instanton part is

$$\begin{aligned} (90) \quad \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}) &:= \mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, 0, \dots, 0) = \\ &= \sum_{(\vec{Y},\vec{v})} \mathbf{q}^{\sum_{\alpha=1}^r n_\alpha + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_\alpha \cdot C \vec{v}_\alpha} \frac{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)} + m) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)} + m)}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} . \end{aligned}$$

Since the *instanton part of the Nekrasov partition function* for $SU(r)$ -gauge theories on \mathbb{R}^4 with one adjoint hypermultiplet of mass m is by [24, Formula (3.26)]

$$\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*, inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}) := \sum_{\vec{Y}} \mathbf{q}^{\sum_{\alpha=1}^r |Y_\alpha|} \prod_{\alpha\beta} \frac{m_{\alpha\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a} + m)}{m_{\alpha\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a})} ,$$

we get

$$(91) \quad \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}) = \sum_{\vec{v}} \mathbf{q}^{\frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}} \frac{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)} + m)}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)})} \cdot \prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*, inst}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}, m; \mathbf{q}) .$$

As before, we define the deformed instanton part of $\mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)})$ as

$$\begin{aligned} \mathcal{Z}_{\vec{v}}^{*def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}) &:= \mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, 0, \dots, 0) = \\ &= \sum_{(\vec{Y}, \vec{v})} \mathbf{q}^{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \vec{v}_{\alpha} \cdot C \vec{v}_{\alpha}} \frac{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)} + m) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)} + m)}{\prod_{\alpha\beta} \prod_{j=1}^{k-1} \ell_{\alpha\beta}^{(j)}(\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \vec{a}^{(j)}) \prod_{i=1}^k m_{\alpha\beta}^{\vec{Y}^i}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)})} \\ &\quad \cdot \prod_{l=1}^k \exp \left(-\tau_1 \left[\text{ch}_{\vec{Y}^l}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)}) \right]_0 \right) . \end{aligned}$$

By using $\mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q})$ we get

$$(92) \quad \mathcal{Z}_{\vec{v}}^{*def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \tau_1) = \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1)^{\frac{1}{k}} \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}_{eff}, \tau_1) ,$$

where the partition function $\mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}_{eff}, \tau_1)$ is defined similarly to (85).

5.4.2. Gauge theory with one adjoint hypermultiplet of mass m . Let us define the *Nekrasov deformed partition function* for $U(r)$ -gauge theories on X_k with one adjoint hypermultiplet of mass m by

$$(93) \quad \mathcal{Z}_{ALE}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\xi}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) := \sum_{\substack{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} i w_i \pmod{k}}} \xi^{\vec{v}} \mathcal{Z}_{\vec{v}}^*(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}, \vec{\tau}, \vec{t}^{(1)}, \dots, \vec{t}^{(k-1)}) ,$$

its *instanton part* by

$$(94) \quad \mathcal{Z}_{ALE}^{\mathcal{N}=2^*, inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\xi}) := \sum_{\substack{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} i w_i \pmod{k}}} \xi^{\vec{v}} \mathcal{Z}_{\vec{v}}^{*inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) ,$$

and its *deformed instanton part* by

$$(95) \quad \mathcal{Z}_{ALE}^{\mathcal{N}=2^*, def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathbf{q}, \vec{\xi}) := \sum_{\substack{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \\ kv_{k-1} \equiv \sum_{i=0}^{k-1} i w_i \pmod{k}}} \xi^{\vec{v}} \mathcal{Z}_{\vec{v}}^{*def-inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}) \\ = \mathcal{Z}_{\mathbb{R}^4}^{cl}(\varepsilon_1, \varepsilon_2, \vec{a})^{\frac{1}{k}} \mathcal{Z}_{ALE}^{\mathcal{N}=2^*, inst}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathbf{q}_{eff}, \vec{\xi}) .$$

CHAPTER 6

AGT conjecture for $U(1)$ -gauge theories on \mathbb{R}^4

This chapter collects part of the literature about the Alday-Gaiotto-Tachikawa conjecture for $U(1)$ -gauge theories on \mathbb{R}^4 , in particular it is dedicated to the proof of the conjecture in the pure and adjoint masses cases. We start in Section 6.1 with some preliminary material such as partitions, associated Young diagrams, Macdonald and Jack symmetric functions and Hilbert schemes of points on surfaces. In Section 6.2 we study (localized) equivariant cohomology of Hilbert schemes of points on \mathbb{C}^2 as a representation of the infinite-dimensional Heisenberg algebra, recalling the famous result by Nakajima and Grojnowski [83, 50], subsequently generalized in various works [105, 75, 98]. In the last section we present the statement of the conjecture and some historical background, then we use the tools developed in the first sections for computing the Nekrasov partition function and completing the proof of the conjecture following [24, 28, 101].

Since in this chapter we will consider only instanton parts of partition functions, we will omit the superscript *inst*, writing $\mathcal{Z}_{\mathbb{R}^4}^\bullet$ for $\mathcal{Z}_{\mathbb{R}^4}^{\bullet, inst}$.

6.1. Preliminaries

In this section we collect some preliminary material we need in this and in next chapter. We sketch a brief introduction to partitions and associated Young diagrams following [78, Chapter I], then we give some results about symmetric functions, in particular Macdonald and Jack symmetric functions, for which again our main reference is [78], in particular Chapter IV. Finally, following [84, Chapter 1] we define Hilbert schemes of points on surfaces and discuss some of their properties.

6.1.1. Partitions. A *partition* of a positive integer n is a nonincreasing sequence of positive numbers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$ such that $|\lambda| := \sum_{i=1}^\ell \lambda_i = n$. We call $\ell = \ell(\lambda)$ the *length* of the partition λ . There is another way to describe a partition λ of n : $\lambda = (1^{m_1} 2^{m_2} \dots)$, where $m_i = \#\{l \in \mathbb{N} \mid \lambda_l = i\}$. Then $\sum_i i \cdot m_i = n$ and $\sum_i m_i = \ell$.

One can associate to a partition λ the a Young diagram defined as $Y_\lambda = \{(a, b) \in \mathbb{N}^2 \mid 1 \leq a \leq l, 1 \leq b \leq \lambda_a\}$. Thus λ_a is the length of the a -th column of Y_λ . We shall identify a partition λ with its Young diagram Y_λ . For a partition λ , the *transpose partition* λ' is the partition whose Young diagram $Y_{\lambda'} := \{(j, i) \in \mathbb{N}^2 \mid (i, j) \in \lambda\}$. We denote by Π the set of all Young diagrams. On Π there is a natural partial ordering called *dominance ordering*: for two partitions μ and λ , we write $\mu \leq \lambda$ if and only if $|\mu| = |\lambda|$ and $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$ for all $i \geq 1$. We write $\mu < \lambda$ if and only if $\mu \leq \lambda$ and $\mu \neq \lambda$. Recall that we defined arm and leg length for a *box* s in a Young diagram Y at the beginning of Section 5.2.

6.1.2. Symmetric Functions. Let F be a field of characteristic zero. We call the *algebra of symmetric polynomials in N variables* the subspace $\Lambda_{F,N}$ of $F[x_1, \dots, x_N]$ invariant under the action of the N -th group of permutations σ_N . We have that $\Lambda_{F,N}$ is a graded ring: $\Lambda_{F,N} = \bigoplus_n \Lambda_{F,N}^n$, where $\Lambda_{F,N}^n$ is the space of homogeneous symmetric polynomials in N variables of degree n (together with the zero polynomial).

For any $M > N$ we have morphisms $\rho_{MN}: \Lambda_{F,M} \rightarrow \Lambda_{F,N}$ mapping the variables x_{N+1}, \dots, x_M to zero. Moreover the morphisms ρ_{MN} preserve the grading, hence we can define $\rho_{MN}^n: \Lambda_{F,M}^n \rightarrow \Lambda_{F,N}^n$; this allows us to define

$$\Lambda_F^n := \varprojlim_N \Lambda_{F,N}^n ,$$

and the *algebra of symmetric functions in infinitely many variables* as $\Lambda_F := \bigoplus_n \Lambda_F^n$. In the following, when no confusion is possible we will denote Λ_F (resp. Λ_F^n) just by Λ (resp. Λ^n).

Now we introduce a basis for Λ . To do this we start by defining a basis in Λ_N . Let $\mu = (\mu_1, \dots, \mu_t)$ be a partition with $t \leq N$, we define the polynomial

$$m_\mu(x_1, \dots, x_N) = \sum_{\tau \in \sigma_N} x_1^{\mu_{\tau(1)}} \dots x_N^{\mu_{\tau(N)}} ,$$

where we set $\mu_j = 0$ for $j = t+1, \dots, N$. The polynomial m_μ is symmetric, moreover the set of all m_μ for all the partitions μ with $|\mu| \leq N$ is a basis of Λ_N . Then the set of all m_μ , for all the partitions μ with $|\mu| \leq N$ and $\sum_i \mu_i = n$, is a basis of Λ_N^n . Since for $M > N \geq t$ we have $\rho_{MN}^n(m_\mu(x_1, \dots, x_M)) = m_\mu(x_1, \dots, x_N)$, by using the definition of inverse limit we can define the *monomial symmetric functions* m_μ . By varying of the partitions μ of n , these functions form a basis for Λ^n .

Now we want to define special families of symmetric functions. Let $n \in \mathbb{N}, n \geq 1$, we define the *elementary symmetric function* e_n as

$$e_n := m_{(1^n)} = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n}$$

and we put $e_0 = 1$. For $\mu = (\mu_1, \dots, \mu_t)$ partition, we set $e_\mu := e_{\mu_1} e_{\mu_2} \dots e_{\mu_t}$: the set $\{e_\mu\}_\mu$ is a basis of Λ . We call *n -th complete symmetric function* the symmetric function $h_n := \sum_{|\nu|=n} m_\nu$. For $\mu = (\mu_1, \dots, \mu_t)$ partition, we set $h_\mu := h_{\mu_1} h_{\mu_2} \dots h_{\mu_t}$: as before, the set $\{h_\mu\}_\mu$ is a basis of Λ . Finally, the *n -th power sum symmetric function* p_n is $p_n := m_{(n)} = \sum_i x_i^n$. As before, the set consisting of symmetric functions $p_\mu := p_{\mu_1} p_{\mu_2} \dots p_{\mu_t}$, for $\mu = (\mu_1, \dots, \mu_t)$ partition, is another basis of Λ .

6.1.2.1. Macdonald functions. Fix a parameter $q \in \mathbb{C}$ with $|q| < 1$. For $a \in \mathbb{C}$, we use throughout the standard hypergeometric notation for the infinite q -shifted factorial

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - a q^n) .$$

We set $F = \mathbb{C}$ throughout and we fix a parameter $t \in \mathbb{C}$ (everything works for any field extension $\mathbb{C} \subseteq F$ and $t \in F$). Define an inner product on the vector space $\Lambda \otimes \mathbb{Q}(q, t)$ such that the basis of power sum symmetric functions $p_\lambda(x)$ are orthogonal with respect to this inner product with the normalization

$$(96) \quad \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} ,$$

where $\delta_{\lambda,\mu} := \prod_i \delta_{\lambda_i,\mu_i}$ and

$$z_\lambda := \prod_{j \geq 1} j^{m_j} m_j! .$$

This is called the *Macdonald inner product*.

DEFINITION 6.1. The monic form of the *Macdonald functions* $M_\lambda(x; q, t) \in \Lambda \otimes \mathbb{Q}(q, t)$ for $x = (x_1, x_2, \dots)$ are uniquely defined by the following two conditions [78, Chapter VI]:

(i) Triangular expansion in the basis $m_\mu(x)$ of monomial symmetric functions:

$$(97) \quad M_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} v_{\lambda,\mu}(q, t) m_\mu(x) \quad \text{with} \quad v_{\lambda,\mu}(q, t) \in \mathbb{C}(q, t) .$$

(ii) Orthogonality:

$$(98) \quad \langle M_\lambda, M_\mu \rangle_{q,t} = \delta_{\lambda,\mu} \prod_{s \in Y_\lambda} \frac{1 - q^{a(s)+1} t^{\ell(s)}}{1 - q^{a(s)} t^{\ell(s)+1}} .$$

■

Note that for $t = 1$ these functions coincide with the monomial symmetric functions, $M_\lambda(x; q, 1) = m_\lambda(x)$. Moreover by their definition the Macdonald functions are homogeneous:

$$(99) \quad M_\lambda(\zeta x; q, t) = \zeta^{|\lambda|} M_\lambda(x; q, t) \quad \text{for} \quad \zeta \in \mathbb{C} .$$

6.1.2.2. Jack functions.

DEFINITION 6.2. Fix $\beta \in \mathbb{C}$, and consider the limit of the Macdonald symmetric functions for $t = p^\beta$ with $p \rightarrow 1$. The resulting symmetric functions are called (*monic*) *Jack function*

$$J_\lambda(x; \beta^{-1}) := \lim_{p \rightarrow 1} M_\lambda(x; p, p^\beta)$$

in $\Lambda \otimes \mathbb{Q}(\beta)$.

■

Taking the limit $p \rightarrow 1$ in the Macdonald inner product (96) $\langle -, - \rangle_{p,p^\beta}$ also yields an inner product $\langle -, - \rangle_\beta$ on $\Lambda \otimes \mathbb{Q}(\beta)$ with

$$(100) \quad \langle p_\lambda, p_\mu \rangle_\beta = \delta_{\lambda,\mu} z_\lambda \beta^{-\ell(\lambda)} ,$$

which is called *Jack inner product*. The orthogonality relation (98) becomes, for the Jack functions,

$$(101) \quad \langle J_\lambda, J_\mu \rangle_\beta = \delta_{\lambda,\mu} \prod_{s \in Y_\lambda} \frac{\beta \ell(s) + a(s) + 1}{\beta (\ell(s) + 1) + a(s)} .$$

The homogeneity property (99) in this case becomes

$$J_\lambda(\zeta y; \beta^{-1}) = \zeta^{|\lambda|} J_\lambda(y; \beta^{-1}) \quad \text{for} \quad \zeta \in \mathbb{C} .$$

REMARK 6.3. The Jack functions can be characterized in a way similar to Definition 6.1. In particular they are uniquely determined by the two conditions

(i) Triangular expansion

$$J_\lambda(x; \beta^{-1}) = m_\lambda(x) + \sum_{\mu < \lambda} \psi_{\lambda,\mu}(q, t) m_\mu(x) \quad \text{with} \quad \psi_{\lambda,\mu}(q, t) \in \mathbb{C}(q, t) .$$

(ii) Orthogonality (101).

△

6.1.3. Hilbert schemes of points on surfaces. For a quasiprojective scheme X , the *Hilbert schemes* Hilb_X^P are defined as the schemes representing the functors

$$\text{Hilb}_X^P: \text{Schemes} \rightarrow \text{Sets}$$

which, for a fixed scheme X and polynomial P , send a scheme S to the set of families of closed subschemes of X parametrized by S , with fixed Hilbert polynomial P . Grothendieck proved in [53] that such schemes exist and, if X is projective, they are projective. Thus on the Hilbert scheme Hilb_X^P there is a universal family \mathcal{Z} such that every family of closed subscheme of X parametrized by S with fixed Hilbert polynomial P is induced by a unique morphism $\phi: S \rightarrow \text{Hilb}_X^P$.

DEFINITION 6.4. Let $n \in \mathbb{N}$. The *Hilbert scheme of n points of X* is the scheme $\text{Hilb}^n(X) := \text{Hilb}_X^P$ corresponding to the constant polynomial $P = n$. ■

There is a well-known description for the generic point of $\text{Hilb}^n(X)$, which explains the name *Hilbert scheme of points*. If $x_1, \dots, x_n \in X$ are n distinct point, $Z = \{x_1, \dots, x_n\} \subset X$ is a closed subscheme, and one can show that $Z \in \text{Hilb}^n(X)$. More generally, points of $\text{Hilb}^n(X)$ can be described as ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\text{length}(\mathcal{O}_X/\mathcal{I}) = n$. Roughly speaking, the Hilbert scheme of n points is “the moduli space of n points in X ”.

Another way of thinking of a space that parametrizes configurations of n points in X is to consider the symmetric product

$$S^n(X) := X \times \dots \times X / \sigma_n,$$

where σ_n is the symmetric group of degree n . This just counts points with multiplicities, forgetting that the scheme structure can be more complicated. In fact, there is a morphism, called *Hilbert-Chow morphism* (see [81, 5.4]), defined by

$$(102) \quad \pi: \mathcal{I} \in \text{Hilb}^n(X) \mapsto \sum_{x \in X} \text{length}(\mathcal{O}_{X,x}/\mathcal{I}_x) [x] \in S^n X,$$

which associates a closed subscheme with its support (with multiplicities) seen as a cycle in X . The Hilbert-Chow morphism is an isomorphism on the locus of closed subschemes supported on n distinct points.

From now on X will be a smooth quasiprojective surface. In this case, Fogarty in [41] proved the following result.

THEOREM 6.5. *If X is a smooth quasiprojective surface, then $\text{Hilb}^n(X)$ is quasiprojective and smooth of dimension $2n$. Moreover, the Hilbert-Chow morphism is a resolution of singularities.*

6.2. Equivariant cohomology of $\text{Hilb}^n(\mathbb{C}^2)$

In the following we shall give a brief survey of results about the equivariant cohomology of $\text{Hilb}^n(\mathbb{C}^2)$ and representation of Heisenberg algebras on this cohomology (cf. [50, 75, 83, 98, 105, 84]).

6.2.1. Equivariant basis. Let us consider the action of the torus $T := (\mathbb{C}^*)^2$ on the affine complex plane \mathbb{C}^2 given by $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$, and the induced T -action on the Hilbert schemes of n -points $\text{Hilb}^n(\mathbb{C}^2)$. Following [37, 84, 83], the T -fixed points of $\text{Hilb}^n(\mathbb{C}^2)$ are zero-dimensional subschemes of \mathbb{C}^2 of length n supported at the origin $0 \in \mathbb{C}^2$ and they correspond to partitions λ of n . We shall denote by Z_λ the fixed point in $\text{Hilb}^n(\mathbb{C}^2)^T$ corresponding to the partition λ of n .

Denote by t_i the T -modules corresponding to the characters $\chi_i: (t_1, t_2) \in T \mapsto t_i \in \mathbb{C}^*$, and by ε_i their first equivariant chern class. Then $H_T^*(pt; \mathbb{C}) = H^*(BT; \mathbb{C}) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$.

As explained in Remark 5.12, Formula (61) gives the equivariant Chern character of the tangent space of $\text{Hilb}^n(\mathbb{C}^2)$ at a fixed point Z_λ :

$$\text{ch}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2)) = \sum_{s \in Y_\lambda} \left(e^{(\ell(s)+1)\varepsilon_1 - a(s)\varepsilon_2} + e^{-\ell(s)\varepsilon_1 + (a(s)+1)\varepsilon_2} \right).$$

Therefore

$$\text{Euler}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2)) = (-1)^n \text{Euler}_+(\lambda) \text{Euler}_-(\lambda),$$

where $\text{Euler}_T(\cdot)$ stands for equivariant Euler class and

$$\begin{aligned} \text{Euler}_+(\lambda) &= \prod_{s \in Y_\lambda} ((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2), \\ \text{Euler}_-(\lambda) &= \prod_{s \in Y_\lambda} (\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2). \end{aligned}$$

Let $i_\lambda: \{Z_\lambda\} \hookrightarrow \text{Hilb}^n(\mathbb{C}^2)$ be the inclusion morphism and define the class

$$[\lambda] := i_{\lambda*}(1) \in H_T^{4n}(\text{Hilb}^n(\mathbb{C}^2)).$$

By projection formula, we get

$$\begin{aligned} [\lambda] \cup [\mu] &= \delta_{\lambda, \mu} \text{Euler}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2))[\lambda] = \\ &= (-1)^n \delta_{\lambda, \mu} \text{Euler}_+(\lambda) \text{Euler}_-(\lambda)[\lambda]. \end{aligned}$$

Denote

$$i_n := \bigoplus_{Z_\lambda \in \text{Hilb}^n(\mathbb{C}^2)^T} i_\lambda: \text{Hilb}^n(\mathbb{C}^2)^T \rightarrow \text{Hilb}^n(\mathbb{C}^2).$$

Let $i_n^!: H_T^*(\text{Hilb}^n(\mathbb{C}^2)^T)' \rightarrow H_T^*(\text{Hilb}^n(\mathbb{C}^2))'$ be the induced Gysin map, where

$$H_T^*(\cdot)' := H_T^*(\cdot) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2)$$

is the *localized equivariant cohomology*. By localization theorem, $i_n^!$ is an isomorphism and the inverse is given by

$$(i_n^!)^{-1}: A \mapsto \left(\frac{i_\lambda^*(A)}{\text{Euler}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2))} \right)_{Z_\lambda \in \text{Hilb}^n(\mathbb{C}^2)^T}.$$

From now on, $\mathbb{H}'_{\mathbb{C}^2, n} := H_T^*(\text{Hilb}^n(\mathbb{C}^2))'$. Define the bilinear form

$$\begin{aligned} (103) \quad \langle \cdot, \cdot \rangle_{\mathbb{H}'_{\mathbb{C}^2, n}}: \mathbb{H}'_{\mathbb{C}^2, n} \times \mathbb{H}'_{\mathbb{C}^2, n} &\rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2), \\ (A, B) &\mapsto (-1)^n p_n^! (i_n^!)^{-1}(A \cup B), \end{aligned}$$

where p_n is the projection of $\text{Hilb}^n(\mathbb{C}^2)^T$ to a point.

REMARK 6.6. Our sign convention in defining the bilinear form is different from the one used, for example, in [84] and in [28]. We chose this convention because, under the isomorphism (108) which will be introduced later, (103) becomes exactly the Jack inner product (100). On the other hand, this produces some changes in the sign, in what follows. Every time we say that a result given here coincide with what is known in the literature, the reader should keep in mind “up to the sign convention we chose”. \triangle

By following [75, Section 2.2], we define the distinguished classes

$$[\alpha_\lambda] = \frac{1}{\text{Euler}_+(\lambda)} [\lambda] \in H_T^{2n}(\text{Hilb}^n(\mathbb{C}^2))' .$$

For λ, μ partitions of n one has

$$\begin{aligned} (104) \quad \langle [\alpha_\lambda], [\alpha_\mu] \rangle_{\mathbb{H}'_{\mathbb{C}^2, n}} &= \delta_{\lambda, \mu} \frac{\text{Euler}_-(\lambda)}{\text{Euler}_+(\lambda)} = \delta_{\lambda, \mu} \prod_{s \in Y_\lambda} \frac{(\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2)}{((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2)} = \\ &= \delta_{\lambda, \mu} \prod_{s \in Y_\lambda} \frac{(\ell(s)\beta + a(s) + 1)}{((\ell(s) + 1)\beta + a(s))} , \end{aligned}$$

where

$$(105) \quad \beta = -\varepsilon_1/\varepsilon_2 .$$

By localization theorem and Formula (104), the classes $[\alpha_\lambda]$ form a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -basis for the vector space $\mathbb{H}'_{\mathbb{C}^2} := \bigoplus_{n \geq 0} \mathbb{H}'_{\mathbb{C}^2, n}$. So the symmetric bilinear form (103) is nondegenerate. The symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{H}'_{\mathbb{C}^2, n}}$ defines a symmetric bilinear form

$$\langle \cdot, \cdot \rangle_{\mathbb{H}'_{\mathbb{C}^2}} : \mathbb{H}'_{\mathbb{C}^2} \times \mathbb{H}'_{\mathbb{C}^2} \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$$

by imposing that $\mathbb{H}'_{\mathbb{C}^2, n_1}$ and $\mathbb{H}'_{\mathbb{C}^2, n_2}$ are orthogonal for $n_1 \neq n_2$. Also $\langle \cdot, \cdot \rangle_{\mathbb{H}'_{\mathbb{C}^2}}$ is nondegenerate.

For $n = 1$, the unique partition of n is $\lambda = (1)$. Let us denote by $[\alpha]$ the class corresponding to $\lambda = (1)$. Then

$$\langle [\alpha], [\alpha] \rangle_{\mathbb{H}'_{\mathbb{C}^2}} = \beta^{-1} .$$

Let us denote by D_x and D_y respectively the x and y -axes of \mathbb{C}^2 . By localization, the corresponding equivariant cohomology classes in $H_T^*(\mathbb{C}^2)'$ are:

$$\begin{aligned} [D_x]_T &= \frac{[0]}{\varepsilon_1} = \frac{[0]}{\text{Euler}_+(1)} = [\alpha] , \\ [D_y]_T &= \frac{[0]}{\varepsilon_2} = \frac{[0]}{\text{Euler}_-(1)} = -\beta[\alpha] . \end{aligned}$$

6.2.2. Heisenberg algebra. Following [84], define

$$D_x(n, i) = \{(Z, Z') \in \text{Hilb}^{n+i}(\mathbb{C}^2) \times \text{Hilb}^n(\mathbb{C}^2) \mid Z' \subset Z, \text{supp}(\mathcal{I}_{Z'}/\mathcal{I}_Z) = \{y\} \subset D_x\} ,$$

where $\mathcal{I}_Z, \mathcal{I}_{Z'}$ are the ideal sheaves corresponding to Z, Z' respectively. Let q_1, q_2 denote the projections of $\text{Hilb}^{n+i}(\mathbb{C}^2) \times \text{Hilb}^n(\mathbb{C}^2)$ to the two factors, respectively. Define the linear operators $\mathfrak{p}_{-i}([D_x]_T) \in \text{End}(\mathbb{H}'_{\mathbb{C}^2})$ by

$$\mathfrak{p}_{-i}([D_x]_T)(A) := q_1^!(q_2^* A \cup [D_x(n, i)]_T) ,$$

for $A \in H_T^*(\text{Hilb}^n(\mathbb{C}^2))'$. We also define $\mathfrak{p}_i([D_x]_T) \in \text{End}(\mathbb{H}'_{\mathbb{C}^2})$ to be the adjoint operator of $\mathfrak{p}_{-i}([D_x]_T)$ with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{H}'_{\mathbb{C}^2}}$ on $\mathbb{H}'_{\mathbb{C}^2}$. Finally put $\mathfrak{p}_0([D_x]_T) = 0$. Note that the class $[D_x]_T$ spans $H_T^*(\mathbb{C}^2)'$ over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, so we can define operators $\mathfrak{p}_i(\eta) \in \text{End}(\mathbb{H}'_{\mathbb{C}^2})$ for every class $\eta \in H_T^*(\mathbb{C}^2)'$. The following result is well-known (see [84, 105, 75]).

THEOREM 6.7. *The linear operators $\mathfrak{p}_i(\eta)$, $i \in \mathbb{Z}$ and $\eta \in H_T^*(\mathbb{C}^2)'$, satisfy the Heisenberg commutation relations:*

$$[\mathfrak{p}_k(\eta_1), \mathfrak{p}_l(\eta_2)] = k\delta_{k,-l}\langle \eta_1, \eta_2 \rangle_{\mathbb{H}'_{\mathbb{C}^2,1}} \text{id} \quad \text{and} \quad [\mathfrak{p}_k(\eta), \text{id}] = 0.$$

Furthermore, $\mathbb{H}'_{\mathbb{C}^2}$ becomes the Fock space of the Heisenberg algebra $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2)}$ with the unit $|0\rangle$ in $H_T^0(\text{Hilb}^0(\mathbb{C}^2))'$ being a highest weight vector.

REMARK 6.8. Since $[D_x]_T = [\alpha]$, we get $\mathfrak{p}_i([\alpha]) = \mathfrak{p}_i([D_x]_T)$. \triangle

From now on, define for $i \in \mathbb{Z} \setminus 0$

$$(106) \quad \mathfrak{p}_i := \mathfrak{p}_i([D_x]_T) \quad \text{thus ,}$$

thus the following commutation relations hold

$$(107) \quad [\mathfrak{p}_{-i}, \mathfrak{p}_i] = i\beta^{-1} \text{id}.$$

Since $[D_x]_T$ generates $H_T^*(\mathbb{C}^2)'$ over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, the operators \mathfrak{p}_i generate $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2)}$.

Let $\lambda = (1^{m_1} 2^{m_2} \dots)$ be a partition. Define $\mathfrak{p}_\lambda := \prod_i \mathfrak{p}_{-i}^{m_i}$. Then

$$\langle \mathfrak{p}_\lambda | 0 \rangle, \mathfrak{p}_\mu | 0 \rangle \rangle_{\mathbb{H}'_{\mathbb{C}^2}} = \delta_{\lambda, \mu} z_\lambda \beta^{-\ell(\lambda)}.$$

Let us denote by Λ' the ring of symmetric functions in infinitely many variables $\Lambda_{\mathbb{C}(\varepsilon_1, \varepsilon_2)}$. Consider on Λ' the Jack inner product (100):

$$\langle p_\lambda, p_\mu \rangle_\beta = \delta_{\lambda, \mu} z_\lambda \beta^{-\ell(\lambda)}.$$

Let $J_\lambda(x; \beta^{-1})$ denote the Jack polynomials of parameter β^{-1} (see Section 6.1.2.2). For the next result we refer to [75, Theorem 3.2] (antidiagonal action, i.e, $t = t_1 = t_2^{-1}$) and to [28, Section 1.5], [28] (arbitrary torus action).

THEOREM 6.9. *There exists a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear isomorphism*

$$(108) \quad \phi: \mathbb{H}'_{\mathbb{C}^2} \longrightarrow \Lambda'$$

preserving bilinear forms such that

$$\phi(\mathfrak{p}_\lambda | 0 \rangle) = p_\lambda(x), \quad \phi([D_x^{(\lambda)}]) = m_\lambda(x), \quad \phi([\alpha_\lambda]) = J_\lambda(x; \beta^{-1}).$$

Moreover, via the isomorphism ϕ , the operators \mathfrak{p}_i acts on Λ' by multiplication for p_{-i} if $i < 0$, and as $i\beta^{-1} \frac{\partial}{\partial p_i}$ if $i > 0$.

6.2.2.1. *Whittaker vectors.* Now we characterize a particular class of Whittaker vectors (see Definition 2.8) which will be useful in studying gauge theories.

PROPOSITION 6.10. *Let $\gamma \in \mathbb{C}(\varepsilon_1, \varepsilon_2)$. In the completed Fock space $\prod_{n \geq 0} \mathbb{H}'_{\mathbb{C}^2, n}$, every vector of the form*

$$G(\gamma) := \exp(\gamma \mathfrak{p}_{-1}) |0\rangle$$

is a Whittaker vector of type χ_γ , where $\chi_\gamma: \mathcal{U}(\mathcal{H}^+) \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by

$$\chi_\gamma(\mathfrak{p}_1) = \gamma \beta^{-1} \quad \text{and} \quad \chi_\gamma(\mathfrak{p}_n) = 0, \quad n > 1.$$

PROOF. The statement follows from the formal expansion

$$G(\gamma) = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} (\mathfrak{p}_{-1})^n |0\rangle$$

with respect to the vector $|0\rangle$, together with the relation $\mathfrak{p}_m |0\rangle = 0$ for $m > 0$ and the identity

$$\mathfrak{p}_m (\mathfrak{p}_{-1})^n = n \beta^{-1} \delta_{m,1} (\mathfrak{p}_{-1})^{n-1} + (\mathfrak{p}_{-1})^n \mathfrak{p}_m$$

for $m \geq 1$. □

6.3. $\mathcal{N} = 2$ $U(1)$ -gauge theory on \mathbb{R}^4

We start this section with a brief review of the history of the AGT conjecture, the Nekrasov partition function and instanton counting on \mathbb{R}^4 , then we give the statement for the $U(1)$ -gauge theories case. Part of the relation was already proved in the last section, here we present the rest of the proof. First we compute the instanton part of Nekrasov partition function for the pure case, showing that it is the norm of a \mathfrak{q} -deformed version of the *Gaiotto state*. Then we use Proposition 6.10 to show that the Gaiotto state is a Whittaker vector. The main references for this part are [84, 24, 86, 40, 101]. Then we focus on the case with adjoint masses, computing the partition function and showing the relation with the *Carlsson-Okounkov vertex operator*. Here the references are [24, 28].

6.3.1. Hystorical background. In this section we briefly give an hystorical overview of the the Alday-Gaiotto-Tachikawa conjecture.

In [5] Alday, Gaiotto and Tachikawa uncovered a relation between two-dimensional conformal field theories (CFT) and a certain class of $\mathcal{N} = 2$ four-dimensional supersymmetric $SU(2)$ quiver gauge theories. In particular, it was argued that the conformal blocks in the Liouville field theory coincide with the instanton parts of the Nekrasov partition function. Further, this relation was generalized [6, 109] to CFTs with affine and $\mathcal{W}(\mathfrak{gl}_r)$ -symmetry. It turned out that the extended $\mathcal{W}(\mathfrak{gl}_r)$ conformal symmetry is related to the instanton counting for the $SU(r)$ gauge group.

This conjecture implies the existence of certain structures on the equivariant cohomology of the moduli space $\mathcal{M}(r, n)$ of framed sheaves on \mathbb{CP}^2 . This was proved by Schiffmann and Vasserot [101], by using a degenerate version of the double affine Hecke algebras, and independently by Maulik and Okounkov [80] by using Yangians. In the following, we shall state the conjecture only in the rank one case.

Recall that for rank 1 the moduli space $\mathcal{M}(r, n)$ is simply the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$.

THEOREM 6.11 (AGT relation for $\mathcal{N} = 2$ $U(1)$ -gauge theories on \mathbb{R}^4). *Let $\mathbb{H}'_{\mathbb{C}^2}$ be the total equivariant cohomology of the Hilbert schemes of points on \mathbb{C}^2 .*

- (1) $\mathbb{H}'_{\mathbb{C}^2}$ is equivalent to the Fock space of an Heisenberg algebra \mathcal{H} :

$$\mathbb{H}'_{\mathbb{C}^2} \cong \mathbb{V}_{Fock}.$$

- (2) (Pure case). The Gaiotto state $G := \sum_{n \geq 0} [\text{Hilb}^n(\mathbb{C}^2)]_T$, in the completed vector space $\hat{\mathbb{H}}'_{\mathbb{C}^2}$, is a Whittaker vector with respect to \mathcal{H} .
- (3) (Adjoint multiplet case). There exists a vertex operator $W(\mathcal{O}_{\mathbb{C}^2}(m), z) \in \text{End}(\mathbb{H}'_{\mathbb{C}^2})[z, z^{-1}]$, depending on the generators of \mathcal{H} , such that the supertrace

$$\text{str } \mathbf{q}^{\mathcal{N}} W(\mathcal{O}_{\mathbb{C}^2}(m), z) = \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*}(\varepsilon_1, \varepsilon_2; \mathbf{q}),$$

where $\mathbf{q}^{\mathcal{N}}$ is the box-counting operator and $\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*}$ is the instanton part of the Nekrasov partition function for $\mathcal{N} = 2$ $U(1)$ -gauge theory on \mathbb{R}^4 with one adjoint hypermultiplet of mass m .

The statement (1) was proved in Theorem 6.7. We shall prove (2) and (3) in the next sections.

6.3.2. Pure $\mathcal{N} = 2$ gauge theory. The instanton part of the Nekrasov partition function for the pure $\mathcal{N} = 2$ $U(1)$ gauge theory on \mathbb{R}^4 is by definition (see [88, 24])

$$\begin{aligned} \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}) &:= \sum_{n \in \mathbb{N}} \mathbf{q}^n \int_{\text{Hilb}^n(\mathbb{C}^2)} [\text{Hilb}^n(\mathbb{C}^2)]_T \\ &= \sum_{n \in \mathbb{N}} (-\mathbf{q})^n \langle [\text{Hilb}^n(\mathbb{C}^2)]_T, [\text{Hilb}^n(\mathbb{C}^2)]_T \rangle_{\mathbb{H}'_{\mathbb{C}^2}}. \end{aligned}$$

By localization theorem we obtain

$$\begin{aligned} \langle [\text{Hilb}^n(\mathbb{C}^2)]_T, [\text{Hilb}^n(\mathbb{C}^2)]_T \rangle_{\mathbb{H}'_{\mathbb{C}^2}} &= \sum_{|\lambda|=n} \frac{(-1)^n}{\text{Euler}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2))} \\ &= \sum_{|\lambda|=n} \prod_{s \in Y_\lambda} \frac{1}{((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2)(\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2)} \\ &= \sum_{|\lambda|=n} \frac{1}{\varepsilon_2^{2n}} \prod_{s \in Y_\lambda} \frac{1}{((\ell(s) + 1)\beta + a(s))(\ell(s)\beta + (a(s) + 1))}, \end{aligned}$$

as in [24, Formula (3.16)].

REMARK 6.12. By [87, Formula (4.7)], there is another well-known expression for the partition function:

$$\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}) = \exp\left(\frac{\mathbf{q}}{\varepsilon_1 \varepsilon_2}\right).$$

△

6.3.2.1. *Gaiotto state.* In [46], Gaiotto considers the inducing state of the (completed) Verma module of the Virasoro algebra. It has the property that it is a Whittaker vector for the Verma module, and the norm of its \mathbf{q} -deformation coincides with the Nekrasov partition function of $SU(2)$ pure $\mathcal{N} = 2$ gauge theory on \mathbb{C}^2 . Below we consider the versions of these vectors for $U(1)$ gauge theory on \mathbb{C}^2 .

Following [101], we define the *Gaiotto state* to be the sum of all fundamental classes

$$G := \sum_{n \geq 0} [\text{Hilb}^n(\mathbb{C}^2)]_T$$

in the completed Fock space $\prod_{n \geq 0} \mathbb{H}'_{\mathbb{C}^2, n}$. We introduce also the \mathbf{q} -deformed *Gaiotto state* as the formal power series

$$G_{\mathbf{q}} := \sum_{n \geq 0} \mathbf{q}^n [\text{Hilb}^n(\mathbb{C}^2)]_T \in \prod_{n \geq 0} \mathbf{q}^n \mathbb{H}'_{\mathbb{C}^2, n}.$$

Consider the bilinear form $\prod_{n \geq 0} \mathbf{q}^n \mathbb{H}'_{\mathbb{C}^2, n} \times \prod_{n \geq 0} \mathbf{q}^n \mathbb{H}_{\mathbb{C}^2, n} \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)[[\mathbf{q}]]$ defined by

$$\begin{aligned} \left\langle \sum_{n \geq 0} \mathbf{q}^n \eta_n, \sum_{n \geq 0} \mathbf{q}^n \nu_n \right\rangle_{\mathbb{H}'_{\mathbb{C}^2, \mathbf{q}}} &:= \sum_{n \geq 0} \mathbf{q}^n \int_{\text{Hilb}^n(\mathbb{C}^2)} \eta_n \cup \nu_n \\ &= \sum_{n \geq 0} (-\mathbf{q})^n \langle \eta_n, \nu_n \rangle_{\mathbb{H}'_{\mathbb{C}^2}}. \end{aligned}$$

It follows immediately that the norm of the \mathbf{q} -deformed Gaiotto state is the instanton part of the Nekrasov partition function for the $\mathcal{N} = 2$ $U(1)$ gauge theory on \mathbb{R}^4 :

$$(109) \quad \langle G_{\mathbf{q}}, G_{\mathbf{q}} \rangle_{\mathbb{H}'_{\mathbb{C}^2, \mathbf{q}}} = \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}).$$

By [101, Proposition 9.3], the Gaiotto state G is a Whittaker vector of type χ_{γ} for some $\gamma \in \mathbb{C}(\varepsilon_1, \varepsilon_2)$ as in Proposition 6.10. Note that there the authors used different conventions than us. Instead of determining γ translating their conventions to ours, we prefer to do it this way: first observe that two Whittaker vectors for the same character χ differs by a multiple of the highest weight vector (see Remark 2.9), in this case $|0\rangle$. Thus $G = G(\gamma) + z|0\rangle$. Taking the scalar product with $|0\rangle$ itself, one has immediately that $z = 0$. For determining γ , we compute by formula (109) and Remark 6.12 the norm of the Gaiotto state:

$$\langle G, G \rangle_{\mathbb{H}'_{\mathbb{C}^2}} = \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q} = 1) = \exp\left(\frac{1}{\varepsilon_1 \varepsilon_2}\right),$$

while by the formal power series expansion,

$$\langle G(\gamma), G(\gamma) \rangle_{\mathbb{H}'_{\mathbb{C}^2}} = \exp(\gamma^2 \beta^{-1}).$$

Thus $\gamma = \sqrt{\frac{\beta}{\varepsilon_1 \varepsilon_2}} = \sqrt{-\frac{1}{\varepsilon_2^2}}$. To sum up, we have the following result.

PROPOSITION 6.13. *The Gaiotto state is a Whittaker vector of type χ , where $\chi: \mathcal{U}(\mathcal{H}^+) \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by*

$$\chi(\mathbf{p}_1) = \sqrt{\frac{1}{\beta \varepsilon_1 \varepsilon_2}} = \sqrt{-\frac{1}{\varepsilon_1^2}} \quad \text{and} \quad \chi(\mathbf{p}_n) = 0, \quad n > 1.$$

6.3.3. $\mathcal{N} = 2^*$ gauge theory. Let $T_m = \mathbb{C}^*$ be an algebraic torus, then $H_{T_m}^*(\text{pt}) = \mathbb{C}[m]$. Following again [88, 24], the instanton part of the Nekrasov partition function for $\mathcal{N} = 2$ gauge theory with one adjoint matter hypermultiplet is

$$\begin{aligned} \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}) &:= \sum_{n \in \mathbb{N}} \mathbf{q}^n \int_{\text{Hilb}^n(\mathbb{C}^2)} E_m(\mathcal{T}_{\text{Hilb}^n(\mathbb{C}^2)}) \\ &= \sum_{n \in \mathbb{N}} (-\mathbf{q})^n \langle [\text{Hilb}^n(\mathbb{C}^2)]_T, E_m(\mathcal{T}_{\text{Hilb}^n(\mathbb{C}^2)}) \rangle_{\mathbb{H}'_{\mathbb{C}^2} \otimes H_{T_m}^*(\text{pt})}, \end{aligned}$$

where $\mathcal{T}_{\text{Hilb}^n(\mathbb{C}^2)}$ is the tangent bundle of $\text{Hilb}^n(\mathbb{C}^2)$, and E_m is defined in Section 5.4.1. By localization theorem,

$$\begin{aligned} \langle [\text{Hilb}^n(\mathbb{C}^2)]_T, E_m(\mathcal{T}_{\text{Hilb}^n(\mathbb{C}^2)}) \rangle_{\mathbb{H}'_{\mathbb{C}^2} \otimes H_{T_m}^*(\text{pt})} &= \\ &= (-1)^n \sum_{|\lambda|=n} \frac{\sum_{j=0}^{2n} (c_j)_T (T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2)) m^{2n-j}}{\text{Euler}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2))} \\ &= (-1)^n \sum_{|\lambda|=n} \prod_{s \in Y_\lambda} \frac{((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2 + m)(\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2 - m)}{((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2)(\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2)}, \end{aligned}$$

as in [24, Formula (3.26)].

6.3.3.1. Carlsson-Okounkov operator. Let us denote by $\mathcal{O}_{\mathbb{C}^2}(m)$ the trivial line bundle of \mathbb{C}^2 with an action of T_m by scaling the fibers¹. Now we define the so-called *Carlsson-Okounkov vertex operator* $W(\mathcal{O}_{\mathbb{C}^2}(m), z)$. In [28], Carlsson and Okounkov define such vertex operator for any smooth quasi-projective surface and any line bundle on it. In this section we shall describe only $W(\mathcal{O}_{\mathbb{C}^2}(m), z)$. We refer to Carlsson and Okounkov's paper for a full description of such kind of vertex operators.

Let $\mathcal{Z} \subset \text{Hilb}^n(\mathbb{C}^2) \times \mathbb{C}^2$ be the *universal* subscheme, whose fiber over a point $Z \in \text{Hilb}^n(\mathbb{C}^2)$ is Z itself. Consider

$$\mathcal{Z}_i := p_{i3}^*(\mathcal{O}_{\mathcal{Z}}) \in K(\text{Hilb}^k(\mathbb{C}^2) \times \text{Hilb}^l(\mathbb{C}^2) \times \mathbb{C}^2) \quad \text{for } i = 1, 2,$$

where p_{i3} is the projection to the i -th and third factors. Define the *virtual vector bundle*

$$\mathcal{E} = p_{12*}((\mathcal{Z}_1^\vee + \mathcal{Z}_2 - \mathcal{Z}_1^\vee \otimes \mathcal{Z}_2) \otimes p_3^*(\mathcal{O}_{\mathbb{C}^2}(m))) \in K(\text{Hilb}^k(\mathbb{C}^2) \times \text{Hilb}^l(\mathbb{C}^2)).$$

Note that the fibers of p_3 intersect the support of \mathcal{Z}_i in finite sets, hence p_{12*} is well-defined. If $(Z, Z') \in \text{Hilb}^k(\mathbb{C}^2)^T \times \text{Hilb}^l(\mathbb{C}^2)^T$, then

$$\mathcal{E}|_{(Z, Z')} = \chi(\mathcal{O}_{\mathbb{C}^2}(m)) - \chi(I_Z, I_{Z'} \otimes \mathcal{O}_{\mathbb{C}^2}(m)),$$

where $\chi(E, F) = \sum_{i=0}^2 \text{Ext}^i(E, F)$ for a pair of coherent sheaves E and F on \mathbb{C}^2 .

Define the operator $W(\mathcal{O}_{\mathbb{C}^2}(m), z) \in \text{End}(\mathbb{H}'_{\mathbb{C}^2}[[z, z^{-1}]])$ by

$$(110) \quad (-1)^l \langle W(\mathcal{O}_{\mathbb{C}^2}(m), z)(A), B \rangle_{\mathbb{H}'_{\mathbb{C}^2}} := z^{l-k} \int_{\text{Hilb}^k(\mathbb{C}^2) \times \text{Hilb}^l(\mathbb{C}^2)} \text{Euler}_T(\mathcal{E}) \cup p_1^*(A) \cup p_2^*(B),$$

where $A \in H_T^*(\text{Hilb}^k(\mathbb{C}^2))'$, $B \in H_T^*(\text{Hilb}^l(\mathbb{C}^2))'$ and p_i is the projection from $\text{Hilb}^k(\mathbb{C}^2) \times \text{Hilb}^l(\mathbb{C}^2)$ to the i -th factor, for $i = 1, 2$.

¹Note that the corresponding action of T_m on the sections of $\mathcal{O}_{\mathbb{C}^2}(m)$ is given by the “inverse”.

Now we would like to compute the trace of the operator $W(\mathcal{O}_{\mathbb{C}^2}(m), z)$. Since the odd cohomology of $\text{Hilb}^n(\mathbb{C}^2)$ is zero (cf. [37]), $\text{Hilb}^n(\mathbb{C}^2)$ has not odd equivariant cohomology². Hence the trace coincides with the usual supertrace taken with respect to the standard \mathbb{Z}_2 -grading in equivariant cohomology. Thus we use “str” to denote indifferently the trace or the supertrace.

Let \mathbf{q}^{L_0} be the “box counting” operator on $\mathbb{H}'_{\mathbb{C}^2}$ such that

$$\mathbf{q}^{L_0}|_{\mathbb{H}'_{\mathbb{C}^2, n}} = \mathbf{q}^n \text{id}.$$

Being $W(\mathcal{O}_{\mathbb{C}^2}(m), z)$ diagonal on the fixed point basis, its trace is given by the sum of its the matrix element over this basis:

$$\text{str } \mathbf{q}^{L_0} W(\mathcal{O}_{\mathbb{C}^2}(m), z) = \sum_{n \in \mathbb{N}} \mathbf{q}^n \sum_{|\lambda|=n} \frac{\langle W(\mathcal{O}_{\mathbb{C}^2}(m), z)([\lambda]), [\lambda] \rangle_{\mathbb{H}'_{\mathbb{C}^2}}}{\langle [\lambda], [\lambda] \rangle}.$$

By Formula (110) and [28, Lemma 6], we obtain

$$\begin{aligned} \text{str } \mathbf{q}^{L_0} W(\mathcal{O}_{\mathbb{C}^2}(m), z) &= \sum_{n \in \mathbb{N}} \mathbf{q}^n \sum_{|\lambda|=n} \frac{\text{Euler}_T(\mathcal{E}_{|(Z_\lambda, Z_\lambda)})}{\text{Euler}_T(T_{Z_\lambda} \text{Hilb}^n(\mathbb{C}^2))} = \\ &= \sum_{n \in \mathbb{N}} \mathbf{q}^n \sum_{|\lambda|=n} \prod_{s \in Y_\lambda} \frac{((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2 + m)(\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2 - m)}{((\ell(s) + 1)\varepsilon_1 - a(s)\varepsilon_2)(\ell(s)\varepsilon_1 - (a(s) + 1)\varepsilon_2)}. \end{aligned}$$

Therefore

$$(111) \quad \text{str } \mathbf{q}^{L_0} W(\mathcal{O}_{\mathbb{C}^2}(m), z) = \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}).$$

Now we would like a description of the operator $W(\mathcal{O}_{\mathbb{C}^2}(m), z)$ in terms of operators \mathbf{p}_i , defined in Formula (106), for $i \in \mathbb{Z} \setminus \{0\}$.

THEOREM 6.14. [28, Theorem 1] *$W(\mathcal{O}_{\mathbb{C}^2}(m), z)$ assume the following form as a vertex operator in the Heisenberg operators:*

$$\begin{aligned} W(\mathcal{O}_{\mathbb{C}^2}(m), z) &= \exp \left(\sum_{i>0} \frac{(-1)^{i-1} z^i}{i} \mathbf{p}_{-i}(\text{Euler}_{T \times T_m}(\mathcal{O}_{\mathbb{C}^2}(m))) \right) \cdot \\ &\quad \cdot \exp \left(- \sum_{i>0} \frac{(-z)^{-i}}{i} \mathbf{p}_i(\text{Euler}_{T \times T_m}(\mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}_{\mathbb{C}^2}(m)^\vee)) \right). \end{aligned}$$

Since in $H_T^*(\mathbb{C}^2)'$ we have $1 = \frac{[\alpha]}{\varepsilon_2} = \frac{[D_x]}{\varepsilon_2}$, we get the vertex operator

$$(112) \quad W(\mathcal{O}_{\mathbb{C}^2}(m), z) = \exp \left(\frac{m}{\varepsilon_2} \sum_{i>0} \frac{(-1)^i z^i}{i} \mathbf{p}_{-i} \right) \exp \left(\frac{\varepsilon_1 + \varepsilon_2 - m}{\varepsilon_2} \sum_{i>0} \frac{(-1)^i z^{-i}}{i} \mathbf{p}_i \right).$$

Using this expression, the commutation relations for the Heisenberg operators (107) and Göttsche’s formula for the Poincaré polynomial of the Hilbert schemes of points [49] one

²It is easy to see that, if a variety has no odd cohomology, then the Leray spectral sequence applied to the Borel model of equivariant cohomology degenerates (see [16]), giving the claim.

obtains, as in [28, Corollary 1]

$$\text{str } \mathbf{q}^{L_0} W(\mathcal{O}_{\mathbb{C}^2}(m), z) = \prod_{n \in \mathbb{N}} (1 - \mathbf{q}^n)^{-\langle \text{Euler}_{T \times T_m}(\mathcal{O}_{\mathbb{C}^2}(m)), \text{Euler}_{T \times T_m}(\mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}_{\mathbb{C}^2}(m)^\vee) \rangle_{H_T^*(\mathbb{C}^2)'} - 1},$$

where $\mathcal{K}_{\mathbb{C}^2}$ is the canonical line bundle of \mathbb{C}^2 . Since

$$\text{Euler}_{T \times T_m}(\mathcal{O}_{\mathbb{C}^2}(m)) = -m \quad \text{and} \quad \text{Euler}_{T \times T_m}(\mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}_{\mathbb{C}^2}(m)^\vee) = m - \varepsilon_1 - \varepsilon_2,$$

by Formula (103) we get $\langle \text{Euler}_{T \times T_m}(\mathcal{O}_{\mathbb{C}^2}(m)), \text{Euler}_{T \times T_m}(\mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}_{\mathbb{C}^2}(m)^\vee) \rangle_{H_T^*(\mathbb{C}^2)'} = m(m - \varepsilon_1 - \varepsilon_2)$. We proved the following.

PROPOSITION 6.15.

$$\mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}) = \prod_{n \in \mathbb{N}} (1 - \mathbf{q}^n)^{m(m - \varepsilon_1 - \varepsilon_2) - 1}.$$

Note that, in the case of antidiagonal torus action, i.e., $\varepsilon_1 = -\varepsilon_2$, this result coincide with [89, Formula (6.12)].

CHAPTER 7

AGT conjecture for $U(1)$ -gauge theories on ALE spaces

In this chapter we state and prove our version of the Alday-Gaiotto-Tachikawa relation for $U(1)$ gauge theories on ALE spaces of type A_{k-1} . The chapter is organized similarly to previous one: in Section 7.1 we consider the moduli spaces studied in Section 5.1 only for the rank one case, and we state the main result of the chapter. Then we discuss the first step of the proof, namely the isomorphism, seen in Section 5.1.2, between moduli spaces of rank one framed sheaves on \mathcal{X}_k and Hilbert schemes of points of X_k , focusing in particular on the induced isomorphism in equivariant cohomology. In Section 7.2 we study the (localized) equivariant cohomology of these Hilbert schemes, focusing on the fixed point basis, the invariant divisors basis, and the decomposition as a tensor product of the equivariant cohomology of \mathbb{C}^2 with a rescaled torus action. Then in Section 7.3 we give a geometric construction of an action of a sum of an Heisenberg algebra and a lattice Heisenberg algebra of type A_{k-1} on the equivariant cohomology of the Hilbert schemes of points. We apply the Frenkel-Kac construction to obtain an action of a sum of an Heisenberg algebra and an affine Kac-Moody algebra of type A_{k-1} on the total equivariant cohomology of the moduli spaces of rank one framed sheaves on \mathcal{X}_k , which turns out to be a basic representation, so that the first statement of the main theorem is proved. In the last Section 7.4 we prove the remaining part of the main theorem proving that the *Gaiotto state* is a Whittaker vector for the pure case, and studying the properties of a Carlsson-Okounkov type vertex operator for the adjoint mass case.

Also in this chapter we will consider only instanton parts of partition functions, so we will omit the superscript *inst*, writing \mathcal{Z}_\bullet for $\mathcal{Z}_\bullet^{inst}$.

7.1. Setting and statement of the result

We start this section by considering the moduli spaces of framed sheaves on \mathcal{X}_k introduced in Section 5.1 just for the rank one case, and, without loss of generality, we fix a trivial framing at infinity. Then we introduce their localized equivariant cohomology, which is the object to study for stating and proving an AGT-type relation for ALE spaces, and the relevant algebras $\mathcal{A}(1, k)$ coming from the CFT counterpart, following [14]. We state our AGT-type relation for X_k , namely, the existence of a (geometric) action of the algebras $\mathcal{A}(1, k)$ on the total equivariant cohomology of the moduli spaces of rank one framed sheaves, which is actually a basic representation. We conclude the section showing how we can reduce to the study of the equivariant cohomology of Hilbert schemes of points on X_k .

7.1.1. Equivariant cohomology of the moduli spaces of rank 1 framed sheaves.

Henceforth we consider the moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ of $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s, \vec{w}})$ -framed sheaves on \mathcal{X}_k for fixed rank $r = 1$, and trivial framing at infinity $\mathcal{F}_\infty^{s, \vec{w}} \simeq \mathcal{O}_{\mathcal{D}_\infty}$, i.e., we are fixing $s = 0$ and $\vec{w} = (1, 0, \dots, 0)$.

REMARK 7.1. Setting the degree of the framing sheaf $\mathcal{F}_\infty^{s,\vec{w}}$ equal to 0 is equivalent, by Formula (57), to taking $s = 0$. Once we set this, it is not restrictive to suppose $\mathcal{F}_\infty^{s,\vec{w}} \simeq \mathcal{O}_{\mathcal{D}_\infty}$: tensoring by \mathcal{R}_i one has an isomorphism

$$\mathcal{M}_{1,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}) \simeq \mathcal{M}_{1,\vec{u}+e_i,\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(0, i)).$$

△

Since in the rank one case the discriminant Δ is simply the second Chern number $\int_{\mathcal{X}_k} c_2 := n$, in the following we denote the moduli spaces as

$$\mathcal{M}_{\mathcal{X}_k}(\gamma, n) := \mathcal{M}_{1,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}),$$

where $\gamma \in \mathfrak{Q}$ fixes the first Chern class as $\vec{u} = -C\gamma$ (see Remark 5.3).

Consider now the equivariant cohomology of the moduli spaces $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$, and set

$$\mathbb{W}'_{\gamma,n} := H_T^*(\mathcal{M}_{\mathcal{X}_k}(\gamma, n)) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2).$$

We endow $\mathbb{W}'_{\gamma,n}$ with the nondegenerate $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -valued bilinear form

$$\langle A, B \rangle_{\mathbb{W}'_{\gamma,n}} := (-1)^n p_n^! (i_n^!)^{-1} (A \cup B),$$

where p_n is the projection from $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$ to a point, and $i_n: \mathcal{M}_{\mathcal{X}_k}(\gamma, n)^T \hookrightarrow \mathcal{M}_{\mathcal{X}_k}(\gamma, n)$ is the inclusion of the fixed locus. Note that, by Remark 4.26, γ varies in the root lattice \mathfrak{Q} . Thus we define the *total* equivariant cohomology

$$\mathbb{W}' := \bigoplus_{\gamma \in \mathfrak{Q}, n \in \mathbb{N}} \mathbb{W}'_{\gamma,n},$$

endowed with the nondegenerate $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -valued bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{W}'}$ induced by the forms $\langle \cdot, \cdot \rangle_{\mathbb{W}'_{\gamma,n}}$.

7.1.2. Statement of the AGT relation. As pointed out in [14] The W -algebras that appear in the formulation of the AGT conjecture for \mathbb{R}^4 (see the Introduction) can be realized as a *conformal limit* of the so-called toroidal algebra. In their attempt to give a formulation of an analog of the AGT relation for ALE spaces of type A_{k-1} , the authors of [14] propose to take a conformal limit, depending on k , of such algebra, which turns out to be, in the rank one case, a sum of an Heisenberg algebra and an affine Kac-Moody algebra of type A_{k-1}

$$\mathcal{A}(1, k) := \mathcal{H} + \widehat{\mathfrak{sl}}_k,$$

where the central elements \mathfrak{c} in \mathcal{H} and $\widehat{\mathfrak{sl}}_k$ are identified. Given a representation $\mathcal{A}(1, k) \rightarrow \text{End}(V)$, we say it is of *level 1* if \mathfrak{c} acts as the identity operator.

Following the known definition of basic representations (see for example Definition 2.13 for the basic representation of $\widehat{\mathfrak{sl}}_k$) we want to give a similar notion for $\mathcal{A}(1, k)$. Note that $\mathcal{A}(1, k)$ inherits a triangular decomposition from the triangular decompositions of \mathcal{H} and $\widehat{\mathfrak{sl}}_k$ (see Formulae (14), (16)). Thus we can introduce in an obvious way $\mathcal{A}(1, k)^\pm$, and of course $\mathcal{U}(\mathcal{A}(1, k))$, $\mathcal{U}(\mathcal{A}(1, k)^\pm)$. So we have also the notions of highest weight vector and highest weight representations.

DEFINITION 7.2. A representation $\mathcal{A}(1, k) \rightarrow \text{End}(V)$ on a vector space V is called a *basic representation* if it is an (irreducible) highest weight representation of level 1. ■

REMARK 7.3. It is easy to see that if V is a basic representation of $\mathcal{A}(1, k)$, then V can be decomposed as a tensor product

$$V \cong \mathcal{F} \otimes V(\Lambda_0)$$

of the Fock space of \mathcal{H} and the basic representation of $\widehat{\mathfrak{sl}}_k$ (see Definition 2.5 and 2.13 for the notations). \triangle

We also introduce the notion of Whittaker vector for representations of $\mathcal{A}(1, k)$, following [30]. Note that the Cartan subalgebra $\mathfrak{h} \subset \mathcal{A}(1, k)$ is isomorphic to the sum (taken identifying the central elements \mathfrak{c}) $\mathcal{H} + \mathcal{H}_\Omega$.

DEFINITION 7.4. Let $\chi: \mathcal{U}(\mathcal{H}^+ + \mathcal{H}_\Omega^+) \rightarrow \mathbb{F}$ be an algebra homomorphism, not identically zero on $\mathcal{H}^+ + \mathcal{H}_\Omega^+$, and let V be a $\mathcal{U}(\mathcal{A}(1, k))$ -module. A non-zero vector $w \in V$ is called a *Whittaker vector of type χ* if $\eta \cdot w = \chi(\eta) w$ for all $\eta \in \mathcal{U}(\mathcal{H}^+ + \mathcal{H}_\Omega^+)$. \blacksquare

Now we can state the AGT relation.

THEOREM 7.5 (AGT relation for $\mathcal{N} = 2$ $U(1)$ gauge theory on X_k). *Given $\gamma \in \Omega$, $n \in \mathbb{N}$, denote by $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$ the moduli space parameterizing isomorphism classes $[(\mathcal{E}, \phi_{\mathcal{E}})]$ of $(\mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty})$ -framed sheaves on \mathcal{X}_k of rank one, first Chern class given by γ and second chern number $\int_{\mathcal{X}_k} c_2(\mathcal{E}) = n$. Denote by $\mathbb{W}'_{\gamma, n}$ the localized equivariant cohomology of $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$, and by \mathbb{W}' the total localized equivariant cohomology. There exists an action of $\mathcal{A}(1, k)$ on \mathbb{W}' such that:*

- (1) \mathbb{W}' is equivalent to the basic representation of $\mathcal{A}(1, k)$.
- (2) **(Pure case).** The Gaiotto state

$$G := \sum_{c \in \Omega, n \in \mathbb{N}} [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \in \prod_{c \in \Omega, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n}$$

in the completed total localized equivariant cohomology $\widehat{\mathbb{W}}' = \prod_{c \in \Omega, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n}$ is a Whittaker vector with respect to this representation.

- (3) **(Adjoint multiplet case).** There exists a Carlsson-Okounkov type vertex operator

$$W(\mathcal{O}_{\mathcal{X}_k}(m), z) \in \text{End}(\mathbb{W}')[[z, z^{-1}]]$$

in the elements of the Cartan subalgebra $\mathcal{H} + \mathcal{H}_\Omega \simeq \mathfrak{h} \subset \mathcal{A}(1, k)$ such that

$$\text{str } \mathbf{q}^{\vec{\mathcal{N}}} \vec{\xi}^\gamma W(\mathcal{O}_{\mathcal{X}_k}(m), z) = \mathcal{Z}_{ALE}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2; \mathbf{q}, \vec{\xi}) .$$

where $\mathbf{q}^{\vec{\mathcal{N}}}$ is the box-counting operator (for $\gamma = 0$ is the usual box-counting operator), $\vec{\xi}^\gamma$ is the operator that counts $\gamma \in \Omega$, and $\mathcal{Z}_{ALE}^{\mathcal{N}=2^}$ is the instanton part of the deformed partition function for $\mathcal{N} = 2^*$ $U(1)$ -gauge theory on X_k .*

7.1.3. Moduli of framed sheaves and Hilbert schemes of points. The proof of the theorem is based on the following considerations. Recall from Section 5.1.2 that the Hilbert scheme of n -points $\text{Hilb}^n(X_k)$ of X_k embeds into $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$: if Z is a point of $\text{Hilb}^n(X_k)$, the coherent sheaf $\mathcal{E} := i_*(I_Z) \otimes \mathcal{R}^{-C\gamma}$ is a rank one torsion-free sheaf on \mathcal{X}_k , trivial along \mathcal{D}_∞ , with first Chern class given by γ and $\int_{\mathcal{X}_k} c_2(\mathcal{E}) = n$.

Therefore Z induces a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ in $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$. This defines an inclusion morphism

$$v_{\gamma, n}: \text{Hilb}^n(X_k) \hookrightarrow \mathcal{M}_{\mathcal{X}_k}(\gamma, n)$$

for every $\gamma \in \mathfrak{Q}$, which is an isomorphism of fine moduli spaces by Proposition 5.10.

Note that the T_t -action on \mathcal{X}_k , whose restriction gives the torus action on X_k , naturally lifts to both $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$ and $\text{Hilb}^n(X_k)$, and the isomorphism described above is equivariant with respect to these actions. Thus we have:

COROLLARY 7.6. *$i_{\gamma, n}$ induces an isomorphism*

$$\bigoplus_{\gamma \in \mathfrak{Q}, n \in \mathbb{N}} i_{\gamma, n}^* : \mathbb{W}' \xrightarrow{\sim} \bigoplus_{\gamma \in \mathfrak{Q}, n \in \mathbb{N}} H_T^*(\text{Hilb}^n(X_k)) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2) \simeq \left(\bigoplus_{n \in \mathbb{N}} H_T^*(\text{Hilb}^n(X_k)) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2) \right) \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q}].$$

Moreover, $i_{\gamma, n}^*$ sends the fundamental class $[\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \in \mathbb{W}'_{\gamma, n}$ to the fundamental class $[\text{Hilb}^n(X_k)]_T \in H_T^*(\text{Hilb}^n(X_k))'$. Therefore

$$[\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \mapsto [\text{Hilb}^n(X_k)]_T \otimes \gamma.$$

7.2. The equivariant cohomology of $\text{Hilb}^n(X_k)$

This section is the most technical part of the proof of Theorem 7.5. Here we study the equivariant cohomology of Hilbert schemes of points on X_k . In particular we are interested in distinguished bases of this cohomology, such as the fixed point basis given by localization theorem and the basis given by torus-invariant divisors. The study of these bases singles out some properties of the equivariant cohomology of the Hilbert schemes of point on X_k , the most important one being the fact that it can be decomposed into a tensor product of equivariant cohomologies of Hilbert schemes of points on the U_i 's.

As we pointed out above, the T_t -action on X_k lifts naturally to a T_t -action to $\text{Hilb}^n(X_k)$. A T_t -fixed subscheme Z of X_k of length n is a disjoint union of T_t -fixed subschemes Z_i , $i = 1, \dots, k$, supported at the T_t -fixed points p_i and $\sum_{i=1}^k \text{length}_{p_i}(Z_i) = n$. Put $n_i = \text{length}_{p_i}(Z_i)$. Since p_i is the T_t -fixed point of the smooth affine surface U_i , as we saw in Section 6.2.1, the T_t -fixed subscheme $Z_i \in \text{Hilb}^{n_i}(U_i)$ corresponds to a Young diagram Y_Z^i of n_i , for $i = 1, \dots, k$. Thus the T_t -fixed point Z corresponds to a k -tuple of Young diagrams $\vec{Y}_Z = (Y_Z^1, \dots, Y_Z^k)$ such that $|\vec{Y}_Z| := \sum_{i=1}^k |Y_Z^i| = n$.

We start with the following result.

LEMMA 7.7. *Let Z be a T_t -fixed point of $\text{Hilb}^n(X_k)$. Then we have the following T_t -equivariant isomorphism*

$$T_Z \text{Hilb}^n(X_k) \simeq \bigoplus_{i=1}^k T_{Z_i} \text{Hilb}^{n_i}(U_i),$$

where $Z = \bigsqcup_{i=1}^k Z_i$ and n_i is the length of Z_i at p_i for $i = 1, \dots, k$.

PROOF. Let \mathcal{O}_{Z_i} be the structure sheaf of Z_i for $i = 1, \dots, k$, then

$$\mathcal{O}_Z = \bigoplus_{i=1}^k \mathcal{O}_{Z_i}.$$

Applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O}_{X_k} & \longrightarrow & \mathcal{O}_Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}_{Z_i} & \longrightarrow & \mathcal{O}_{X_k} & \longrightarrow & \mathcal{O}_{Z_i} \longrightarrow 0 \end{array}$$

we get $\mathcal{I}_{Z_i}/\mathcal{I}_Z = \ker(\mathcal{O}_Z \rightarrow \mathcal{O}_{Z_i}) = \bigoplus_{\substack{1 \leq j \leq k \\ j \neq i}} \mathcal{O}_{Z_j}$, hence $\mathcal{I}_{Z_i}/\mathcal{I}_Z$ is a coherent sheaf supported

only at the fixed points p_j for $j \in \{1, \dots, k\}, j \neq i$. In particular, the stalk at p_i of the inclusion morphism $\mathcal{I}_Z \hookrightarrow \mathcal{I}_{Z_i}$ is the identity. On the other hand, as described in [84, Chapter 1], we have

$$T_Z \text{Hilb}^n(X_k) \simeq \text{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \simeq \bigoplus_{i=1}^k \text{Hom}(\mathcal{I}_Z, \mathcal{O}_{Z_i}).$$

Note that

$$\text{Hom}(\mathcal{I}_Z, \mathcal{O}_{Z_i}) = \text{Hom}((\mathcal{I}_Z)_{p_i}, (\mathcal{O}_{Z_i})_{p_i}) = \text{Hom}((\mathcal{I}_{Z_i})_{p_i}, (\mathcal{O}_{Z_i})_{p_i}).$$

Therefore

$$T_Z \text{Hilb}^n(X_k) \simeq \bigoplus_{i=1}^k T_{Z_i} \text{Hilb}^{n_i}(U_i).$$

It remains to show that this isomorphism is T_t -equivariant. First, note that all subvarieties of X_k considered, Z and Z_i for $i = 1, \dots, k$, are T_t -invariant. The stalks of their sheaves of ideals \mathcal{I}_Z and \mathcal{I}_{Z_i} are ideals in $\mathcal{O}_{X_k, p}$ generated by monomials, hence they are equivariant ideals (cf. [97, Section 5]). Then they are T_t -equivariant and the same for $\text{Hom}(\mathcal{I}_Z, \mathcal{O}_{Z_i})$ and $\text{Hom}(\mathcal{I}_{Z_i}, \mathcal{O}_{Z_i})$, which we have shown to be isomorphic. The isomorphism is T_t -equivariant, being actually just the identity between the unique nonvanishing stalk of the two sheaves, and so it is the isomorphism between the global sections of these, which is our isomorphism. \square

By the previous lemma, we get

$$\text{ch}_T(T_Z \text{Hilb}^n(X_k)) = \sum_{i=1}^k \text{ch}_T(T_{Z_i} \text{Hilb}^{n_i}(U_i)).$$

Recall that the zero-subscheme Z_i corresponds to a Young diagram Y_Z^i for $i = 1, \dots, k$. By using the description (32) of the coordinates ring $\mathbb{C}[U_i]$ of U_i , one can get

$$\text{ch}_T(T_{Z_i} \text{Hilb}^{n_i}(U_i)) = \sum_{s \in Y_Z^i} \left(e^{(\ell(s)+1)\varepsilon_1^{(i)} - a(s)\varepsilon_2^{(i)}} + e^{-\ell(s)\varepsilon_1^{(i)} + (a(s)+1)\varepsilon_2^{(i)}} \right),$$

where $\varepsilon_1^{(i)}, \varepsilon_2^{(i)}$ are defined in Section 5.2.2. From now on, we identify a torus-fixed point Z of $\text{Hilb}^n(X_k)$ with its k -tuple \vec{Y}_Z of Young diagrams.

REMARK 7.8. The character $\text{ch}_T(T_Z \text{Hilb}^n(X_k))$ coincides with the one computed in Section 5.2.2 for the tangent to the moduli space $\mathcal{M}_{\mathcal{X}_k}(\gamma, n)$, under the isomorphism of Proposition 5.10.

\triangle

Let $\vec{Y} = (Y^1, \dots, Y^k)$ be a torus-fixed point. Define

$$\begin{aligned} \text{Euler}_+(\vec{Y}) &:= \prod_{i=1}^k \prod_{s \in Y^i} \left((\ell(s) + 1) \varepsilon_1^{(i)} - a(s) \varepsilon_2^{(i)} \right) \\ \text{Euler}_-(\vec{Y}) &:= \prod_{i=1}^k \prod_{s \in Y^i} \left(\ell(s) \varepsilon_1^{(i)} - (a(s) + 1) \varepsilon_2^{(i)} \right) \end{aligned}$$

Thus the equivariant Euler class of the tangent bundle at the fixed point \vec{Y} is

$$\text{Euler}_T(T_{\vec{Y}} \text{Hilb}^n(X_k)) = (-1)^n \text{Euler}_+(\vec{Y}) \text{Euler}_-(\vec{Y}).$$

7.2.1. Equivariant basis I: Torus-fixed points. Let \vec{Y} be a k -tuple of Young diagrams corresponding to a fixed point in $\text{Hilb}^n(X_k)$. Consider the inclusion morphism $i_{\vec{Y}}: \{\vec{Y}\} \hookrightarrow \text{Hilb}^n(X_k)$ and define the class

$$[\vec{Y}] := (i_{\vec{Y}})_*(1) \in H_T^{4n}(\text{Hilb}^n(X_k)).$$

By the projection formula, we get

$$\begin{aligned} [\vec{Y}] \cup [\vec{Y}'] &= \delta_{\vec{Y}, \vec{Y}'} \text{Euler}_T(T_{\vec{Y}} \text{Hilb}^n(X_k)) [Y] = \\ &= (-1)^n \delta_{\vec{Y}, \vec{Y}'} \text{Euler}_+(\vec{Y}) \text{Euler}_-(\vec{Y}) [\vec{Y}]. \end{aligned}$$

Denote

$$i_n := \bigoplus_{\vec{Y} \in \text{Hilb}^n(X_k)^T} i_{\vec{Y}}: \text{Hilb}^n(X_k)^T \rightarrow \text{Hilb}^n(X_k).$$

Let $i_n^!: H_T^*(\text{Hilb}^n(X_k)^T)' \rightarrow H_T^*(\text{Hilb}^n(X_k))'$ be the induced Gysin map, where $H_T^*(\cdot)' = H_T^*(\cdot) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is the localized equivariant cohomology. By the localization theorem, $i_n^!$ is an isomorphism, and the inverse is given by

$$(i_n^!)^{-1}: \alpha \mapsto \left(\frac{i_{\vec{Y}}^*(\alpha)}{\text{Euler}_T(T_{\vec{Y}} \text{Hilb}^n(X_k))} \right)_{\vec{Y} \in \text{Hilb}^n(X_k)^T}.$$

From now on, $\mathbb{H}'_n := H_T^*(\text{Hilb}^n(X_k))'$. As in Formula (103), define the bilinear form

$$(113) \quad \begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{H}'_n}: \mathbb{H}'_n \times \mathbb{H}'_n &\rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2), \\ (A, B) &\mapsto (-1)^n p_n^! (i_n^!)^{-1} (A \cup B), \end{aligned}$$

where p_n is the projection of $\text{Hilb}^n(X_k)^T$ to a point. As in Section 6.2.1, for any class $[\vec{Y}] \in H_T^{4n}(\text{Hilb}^n(X_k))$, we define a distinguished class

$$[\alpha_{\vec{Y}}] := \frac{[\vec{Y}]}{\text{Euler}_+(\vec{Y})} \in H_T^{2n}(\text{Hilb}^n(X_k))'.$$

Then, by the same computation as in Formula (104), we have

$$\begin{aligned}
 \langle [\alpha_{\vec{Y}}], [\alpha_{\vec{Y}'}] \rangle_{\mathbb{H}'_n} &= \delta_{\vec{Y}, \vec{Y}'} \frac{\text{Euler}_-(\vec{Y})}{\text{Euler}_+(\vec{Y})} \\
 (114) \qquad &= \delta_{\vec{Y}, \vec{Y}'} \prod_{i=1}^k \prod_{s \in Y^i} \frac{\ell(s)\varepsilon_1^{(i)} - (a(s) + 1)\varepsilon_2^{(i)}}{(\ell(s) + 1)\varepsilon_1^{(i)} - a(s)\varepsilon_2^{(i)}} \\
 &= \delta_{\vec{Y}, \vec{Y}'} \prod_{i=1}^k \prod_{s \in Y^i} \frac{\ell(s)\beta_i + a(s) + 1}{(\ell(s) + 1)\beta_i + a(s)},
 \end{aligned}$$

where we defined, as in (105)

$$(115) \qquad \beta_i := -\frac{\varepsilon_1^{(i)}}{\varepsilon_2^{(i)}}.$$

Note that when $n = 1$, \vec{Y} is just a fixed point $p_i \in X_k^T$ for $i = 1, \dots, k$. Thus we have

$$\begin{aligned}
 \text{Euler}_+(p_i) &= \varepsilon_1^{(i)} = (k - i + 1)\varepsilon_1 + (1 - i)\varepsilon_2, \\
 \text{Euler}_-(p_i) &= -\varepsilon_2^{(i)} = (k - i)\varepsilon_1 - i\varepsilon_2.
 \end{aligned}$$

Therefore

$$\beta_i := \frac{\text{Euler}_+(p_i)}{\text{Euler}_-(p_i)}.$$

If for $i = 1, \dots, k$ we define $[\alpha_i] := [\alpha_{p_i}]$, we get

$$\langle [\alpha_i], [\alpha_j] \rangle_{\mathbb{H}'_1} = \beta_i^{-1} \delta_{i,j} \in \mathbb{C}(\varepsilon_1, \varepsilon_2).$$

By the localization theorem and Formula (114), the classes $[\alpha_{\vec{Y}}]$, where $|\vec{Y}| = n$, form a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear basis of \mathbb{H}'_n . So the bilinear form (113) is nondegenerate; it extends to give a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{H}'}$ on $\mathbb{H}' := \bigoplus_{n \geq 0} \mathbb{H}'_n$, which we shall call *the total equivariant cohomology* of the Hilbert schemes of points on X_k . Note that we can restate the Corollary 7.6 in the following way: the isomorphisms of schemes $i_{\gamma, n}$ induce an isomorphism

$$(116) \qquad \bigoplus_{\gamma, n} i_{\gamma, n}^* : \mathbb{W}' \xrightarrow{\sim} \mathbb{H}' \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q}].$$

Let $i \in \{1, \dots, k\}$. Let $\mathbb{H}(i)'$ be the linear $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -subspace of \mathbb{H}' generated by all the classes $[\vec{Y}]$ associated to fixed points $\vec{Y} = (Y^1, \dots, Y^k)$ such that $Y^j = 0$ for every $j \in \{1, \dots, k\}$, $j \neq i$. First note that, by the localization theorem

$$(117) \qquad \mathbb{H}(i)' \cong \bigoplus_{m \geq 0} H_T^*(\text{Hilb}^m(U_i)) \otimes_{\mathbb{C}[\varepsilon_1^{(i)}, \varepsilon_2^{(i)}]} \mathbb{C}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}).$$

We point out that $\mathbb{C}[\varepsilon_1^{(i)}, \varepsilon_2^{(i)}] = \mathbb{C}[\varepsilon_1, \varepsilon_2]$ and also $\mathbb{C}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}) = \mathbb{C}(\varepsilon_1, \varepsilon_2)$. Thus we can define, as we did for \mathbb{C}^2

$$\begin{aligned}
 \mathbb{H}'_{U_i, m} &:= H_T^*(\text{Hilb}^m(U_i)) \otimes_{\mathbb{C}[\varepsilon_1^{(i)}, \varepsilon_2^{(i)}]} \mathbb{C}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}) = H_T^*(\text{Hilb}^m(U_i)) \otimes_{\mathbb{C}[\varepsilon_1, \varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2) \\
 \mathbb{H}'_{U_i} &:= \bigoplus_{m \geq 0} \mathbb{H}'_{U_i, m},
 \end{aligned}$$

and rewrite the previous isomorphism as $\mathbb{H}(i)' \cong \mathbb{H}'_{U_i}$. Again by the localization theorem, there exists a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear isomorphism

$$(118) \quad \Omega: \mathbb{H}' \xrightarrow{\sim} \bigotimes_{i=1}^k \mathbb{H}(i)' \xrightarrow{\sim} \bigotimes_{i=1}^k \mathbb{H}'_{U_i} .$$

In particular, for a fixed point $\vec{Y} = (Y^1, \dots, Y^k)$ we get:

$$\Omega: [\alpha_{\vec{Y}}] \mapsto [\alpha_{Y^1}] \otimes \dots \otimes [\alpha_{Y^k}] .$$

Moreover, the isomorphism Ω intertwines the bilinear forms $\langle \cdot, \cdot \rangle_{\mathbb{H}'}$ and $\prod_{i=1}^k \langle \cdot, \cdot \rangle_i$, where $\langle \cdot, \cdot \rangle_i$ is the symmetric bilinear form on \mathbb{H}'_{U_i} analog to (103). In a similar way, we have a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear isomorphism

$$(119) \quad \Omega_k: H_T^*(X_k)' \xrightarrow{\sim} \bigoplus_{j=1}^k \bigotimes_{i=1}^k \mathbb{H}'_{U_i, \delta_{i,j}} \xrightarrow{\sim} \bigoplus_{i=1}^k \mathbb{H}'_{U_i, 1} .$$

In this case, $\Omega_k: [\alpha_i] \mapsto (|0\rangle_{\mathbb{H}'_{U_1}}, \dots, [\alpha_i], \dots, |0\rangle_{\mathbb{H}'_{U_k}})$, where the class $[\alpha_i]$ on the left-hand side belongs to $H_T^*(X_k)'$ while on the right-hand side it belongs to $\mathbb{H}'_{U_i, 1}$ as defined in Section 6.2.1; moreover we denote by $|0\rangle_{\mathbb{H}'_{U_i}}$ the unit in $\mathbb{H}'_{U_i, 0}$. As before, Ω_k intertwines the symmetric bilinear forms.

7.2.2. Equivariant basis II: Torus-invariant divisors. Let $[D_i]_T$ be the class in \mathbb{H}'_1 given by the T_t -invariant divisor D_i for $i = 0, \dots, k$. For $i = 1, \dots, k-1$,

$$(120) \quad \begin{aligned} [D_i]_T &= \frac{[p_i]}{\text{Euler}_T(T_{p_i} D_i)} + \frac{[p_{i+1}]}{\text{Euler}_T(T_{p_{i+1}} D_i)} = \frac{[p_i]}{\varepsilon_2^{(i)}} + \frac{[p_{i+1}]}{\varepsilon_1^{(i+1)}} = \\ &= -\beta_i[\alpha_i] + [\alpha_{i+1}] . \end{aligned}$$

Thus we obtain for $i, j = 1, \dots, k-1$

$$(121) \quad \langle [D_i]_T, [D_j]_T \rangle_{\mathbb{H}'_1} = \begin{cases} 2 & i = j , \\ -1 & |i - j| = 1 , \\ 0 & \text{otherwise} . \end{cases}$$

Moreover, by applying the localization theorem to $[D_0]_T$ and $[D_k]_T$, we obtain

$$\begin{aligned} [D_0]_T &= \frac{[p_1]}{k\varepsilon_1} = \frac{[p_1]}{\varepsilon_1^{(1)}} = [\alpha_1] , \\ [D_k]_T &= \frac{[p_k]}{k\varepsilon_2} = \frac{[p_k]}{\varepsilon_2^{(k)}} = -\beta_k[\alpha_k] . \end{aligned}$$

By using these formulas, one can straightforward obtain

$$\langle [D_0]_T, [D_i]_T \rangle_{\mathbb{H}'_1} = \begin{cases} \beta_1^{-1} & i = 0 , \\ -1 & i = 1 , \\ 0 & \text{otherwise} , \end{cases}$$

and

$$\langle [D_k]_T, [D_i]_T \rangle_{\mathbb{H}'_1} = \begin{cases} \beta_k & i = k , \\ -1 & i = k - 1 , \\ 0 & \text{otherwise} . \end{cases}$$

Now we can relate the classes $[\alpha_i]$, for $i = 1, \dots, k$, to the classes $[D_j]_T$, for $j = 0, \dots, k$. By using Formula (120), one obtain for $i = 2, \dots, k$

$$(122) \quad [\alpha_i] = \sum_{j=0}^{i-2} \left(\prod_{s=j+1}^{i-1} \beta_s \right) [D_j]_T + [D_{i-1}]_T .$$

Since $\text{Euler}_+(p_l) = \text{Euler}_-(p_{l-1})$ for $l = 2, \dots, k$, we get $\prod_{s=j+1}^{i-1} \beta_s = \frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_-(p_{i-1})}$. By using the definition of $[\alpha_k]$ and Formula (122) for $i = k$ we obtain

$$-\beta_k^{-1}[D_k]_T = [\alpha_k] = \sum_{j=0}^{k-1} \frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_-(p_{k-1})} [D_j]_T .$$

If we put formally $\text{Euler}_+(p_{k+1}) := \text{Euler}_-(p_k)$, we can reformulate the previous formula as

$$(123) \quad \sum_{j=0}^k \text{Euler}_+(p_{j+1}) [D_j]_T = 0 ,$$

and in particular for all $i = 0, \dots, k$ we have $[D_i]_T = - \sum_{\substack{0 \leq j \leq k \\ j \neq i}} \left(\frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_+(p_{i+1})} \right) [D_j]_T$. As we saw previously, the classes $\alpha_1, \dots, \alpha_k$ form a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear basis of \mathbb{H}'_1 . By Relations (122) and (123), also

$$(124) \quad \{[D_0]_T, [D_1]_T, \dots, [D_{k-1}]_T\} \quad \text{and} \quad \{[D_1]_T, [D_2]_T, \dots, [D_k]_T\}$$

are $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear bases in \mathbb{H}'_1 . Moreover, with respect to the isomorphism Ω_k defined in Formula (119), we have for $i = 1, \dots, k-1$

$$\Omega_k: [D_i]_T \mapsto -\beta_i(|0\rangle \otimes \dots \otimes [\alpha_i] \otimes \dots \otimes |0\rangle) + |0\rangle \otimes \dots \otimes [\alpha_{i+1}] \otimes \dots \otimes |0\rangle ,$$

and a similar description for $[D_0]_T$ and $[D_k]_T$.

7.3. The basic representation of $\mathcal{A}(1, k)$

This section is the representation-theoretical part of the proof of Theorem 7.5. Here we construct Nakajima-type operators on the equivariant cohomology of $\text{Hilb}^n(X_k)$, obtaining an irreducible highest weight representation of a rank k Heisenberg algebra (see Example 2.3) on this equivariant cohomology. By this we obtain also an irreducible highest weight representation of a sum of an Heisenberg algebra and a lattice Heisenberg algebra of type A_{k-1} , on the same equivariant cohomology. Then we apply the Frenkel-Kac construction to this representation, obtaining a basic representation (cf. Definition 7.2), of the algebra $\mathcal{A}(1, k)$ via the isomorphism (116). Thanks to this we prove the first statement in (1) of Theorem 7.5. Finally we give a characterization of a certain class of Whittaker vectors for this representation, which will be useful in the next section.

We start with the construction of the Nakajima operators. Let i be a positive integer and Y a torus-invariant closed curve in X_k . Define

$$Y_{n,i} := \{(Z, Z') \in \text{Hilb}^{n+i}(X_k) \times \text{Hilb}^n(X_k) \mid Z' \subset Z, \text{Supp}(\mathcal{I}_{Z'}/\mathcal{I}_Z) = \{y\} \subset Y\}.$$

Let q_1 and q_2 be the projections of $\text{Hilb}^{n+i}(X_k) \times \text{Hilb}^n(X_k)$ to the two factors respectively. We define the linear operator $\mathbf{p}_{-i}([Y]): \mathbb{H}' \rightarrow \mathbb{H}'$ which acts on $\alpha \in \mathbb{H}'_n$ as $\mathbf{p}_{-i}([Y])(\alpha) :=$

$q_1^1(q_2^*(\alpha) \cup [Y_{n,i}]) \in \mathbb{H}'_{n+i}$. This definition is well-posed because the restriction of q_1 to $Y_{n,i}$ is proper. Since the bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{H}'}$ is nondegenerate on \mathbb{H}' , we define $\mathfrak{p}_i([Y])$ to be the adjoint operator of $\mathfrak{p}_{-i}([Y])$. Finally, put $\mathfrak{p}_0([Y]) = 0$. By using one of the two bases in (124), we extend by linearity on α to obtain the linear operator $\mathfrak{p}_i(\alpha)$ for every $\alpha \in \mathbb{H}'_1 = H_T^*(X_k)'$.

THEOREM 7.9. *The linear operators $\mathfrak{p}_m(\alpha)$, where $m \in \mathbb{Z}$ and $\alpha \in H_T^*(X_k)'$, satisfy the Heisenberg commutation relation:*

$$[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)] = m\delta_{m,-n}\langle \alpha, \beta \rangle_{\mathbb{H}'_1} \text{id} \quad \text{and} \quad [\mathfrak{p}_m(\alpha), \text{id}] = 0.$$

Furthermore \mathbb{H}' is the Fock space of the Heisenberg algebra $\mathcal{H}_{\mathbb{H}'_1}$ modeled on $\mathbb{H}'_1 = H_T^*(X_k)'$ with highest weight vector $|0\rangle_{\mathbb{H}'}$ the unit element in $H_T^0(\text{Hilb}^0(X_k))'$.

PROOF. Let us fix the $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear basis $\{[D_0]_T, [D_1]_T, \dots, [D_{k-1}]_T\}$ in \mathbb{H}'_1 . Since $\mathfrak{p}_m([D_i]_T)$ is the adjoint operator of $\mathfrak{p}_{-m}([D_i]_T)$ for $i = 1, \dots, k$, we need only to prove that

$$(125) \quad [\mathfrak{p}_m([D_i]_T), \mathfrak{p}_n([D_j]_T)] = 0,$$

$$(126) \quad [\mathfrak{p}_m([D_i]_T), \mathfrak{p}_{-n}([D_j]_T)] = m\delta_{m,n}\langle [D_i]_T, [D_j]_T \rangle \text{id},$$

for $m, n > 0$ and $0 \leq i, j \leq k$. When $i \neq j$, D_i and D_j are either disjoint or intersect transversally at exactly one point. Following the argument in [84, 105] we conclude that Formulas (125) and (126) hold for $i \neq j$. For $n, m > 0$, $0 \leq i \leq k$, we have

$$\begin{aligned} [\mathfrak{p}_m([D_i]_T), \mathfrak{p}_n([D_i]_T)] &= [\mathfrak{p}_m([D_i]_T), -\sum_{\substack{0 \leq j \leq k \\ j \neq i}} \frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_+(p_{i+1})} \mathfrak{p}_n[D_j]_T] \\ &= -\sum_{\substack{0 \leq j \leq k \\ j \neq i}} \frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_+(p_{i+1})} [\mathfrak{p}_m([D_i]_T), \mathfrak{p}_n([D_j]_T)] = 0. \end{aligned}$$

For $n, m > 0$, $0 < i < k$, we get

$$\begin{aligned} [\mathfrak{p}_m([D_i]_T), \mathfrak{p}_{-n}([D_i]_T)] &= -\sum_{\substack{0 \leq j \leq k \\ j \neq i}} \frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_+(p_{i+1})} [\mathfrak{p}_m([D_i]_T), \mathfrak{p}_{-n}([D_j]_T)] \\ &= -\sum_{\substack{0 \leq j \leq k \\ j \neq i}} \frac{\text{Euler}_+(p_{j+1})}{\text{Euler}_+(p_{i+1})} m\delta_{m,n}\langle [D_i]_T, [D_j]_T \rangle \text{id} \\ &= \left(\frac{\text{Euler}_+(p_i) + \text{Euler}_+(p_{i+2})}{\text{Euler}_+(p_{i+1})} \right) m\delta_{m,n} \text{id} \\ &= 2m\delta_{m,n} \text{id} = m\delta_{m,n}\langle [D_i]_T, [D_i]_T \rangle \text{id}. \end{aligned}$$

In a similar way, for $i = 0, k$ we have

$$\begin{aligned} [\mathfrak{p}_m([D_0]_T), \mathfrak{p}_{-n}([D_0]_T)] &= \beta_1^{-1} m\delta_{m,n} \text{id} = m\delta_{m,n}\langle [D_0]_T, [D_0]_T \rangle \text{id}, \\ [\mathfrak{p}_m([D_k]_T), \mathfrak{p}_{-n}([D_k]_T)] &= \beta_k m\delta_{m,n} \text{id} = m\delta_{m,n}\langle [D_k]_T, [D_k]_T \rangle \text{id}. \end{aligned}$$

To prove the second statement, recall that the classes $\alpha_{\vec{Y}}$ form a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear basis of \mathbb{H}'_n for $|\vec{Y}| = n$. Therefore, as in the antidiagonal case described in [98, Formula 2.27], we obtain

$$(127) \quad \sum_{n=0}^{\infty} \dim_{\mathbb{C}(\varepsilon_1, \varepsilon_2)}(\mathbb{H}'_n) q^n = \sum_{n=0}^{\infty} \sum_{|\vec{Y}|=n} q^n = \prod_{l=1}^{\infty} \frac{1}{(1 - q^l)^k}.$$

Hence we can identify the space \mathbb{H}' with the Fock space of the Heisenberg algebra. \square

7.3.1. The Heisenberg algebra of rank k . Let $i \in \{1, \dots, k\}$. Consider the Heisenberg algebra \mathcal{H}_i over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ generated by the operators

$$\mathfrak{p}_{-m}^i := \mathfrak{p}_{-m}(\alpha_i) \quad \text{and} \quad \mathfrak{p}_m^i := \mathfrak{p}_m(\alpha_i)$$

for $m \in \mathbb{N}_{>0}$. By Theorem 7.9, we have the following commutation relations

$$(128) \quad [\mathfrak{p}_m^i, \mathfrak{p}_n^i] = m\delta_{m,-n} \langle \alpha_i, \alpha_i \rangle_{\mathbb{H}'} \text{id} = m\delta_{m,-n} \beta_i^{-1} \text{id}.$$

Let $\mathbb{H}(i)'$ be the linear $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -subspace of \mathbb{H}' introduced in Section 7.2.1. Then by Theorem 6.7 $\mathbb{H}(i)'$ is the Fock space for the Heisenberg algebra \mathcal{H}_i , for any $i \in \{1, \dots, k\}$; therefore, the $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -vector space $\mathbb{H}(i)'$ is generated by the elements $\mathfrak{p}_\lambda^i |0\rangle$, where $\mathfrak{p}_\lambda^i := \prod_{l \geq 1} (\mathfrak{p}_{-l}^i)^{m_l}$ for $\lambda = (1^{m_1} 2^{m_2} \dots)$ partition. One can show that

$$\langle \mathfrak{p}_\lambda^i |0\rangle, \mathfrak{p}_\mu^i |0\rangle \rangle_{\mathbb{H}(i)'} = \delta_{\lambda, \mu} z_\lambda \beta_i^{-\ell(\lambda)}.$$

On the algebra $\Lambda' := \Lambda_{\mathbb{C}(\varepsilon_1, \varepsilon_2)}$ of symmetric functions over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ we define the Jack inner product (100) with parameter β_i :

$$\langle p_\lambda, p_\mu \rangle_{\beta_i} := \delta_{\lambda, \mu} z_\lambda \beta_i^{-\ell(\lambda)}.$$

We shall denote with Λ'_{β_i} the algebra Λ' endowed with the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\beta_i}$. Thus by the isomorphism (117) and Theorem 6.9 there exists an isomorphism of $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -vector spaces

$$\begin{aligned} \Phi_i: \mathbb{H}(i)' &\xrightarrow{\sim} \Lambda'_{\beta_i}, \\ \mathfrak{p}_\lambda^i |0\rangle &\longmapsto p_\lambda, \end{aligned}$$

which intertwines the symmetric bilinear forms $\langle \cdot, \cdot \rangle_{\mathbb{H}(i)'}$ and $\langle \cdot, \cdot \rangle_{\beta_i}$. Moreover, the operator \mathfrak{p}_{-n}^i acts by multiplication by p_n on Λ'_{β_i} and the operator \mathfrak{p}_n^i , being the adjoint with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{H}(i)'}$, acts as $n\beta_i^{-1} \frac{\partial}{\partial p_n}$.

By Theorem 6.9 we can also determine how Φ_i acts on the $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -basis $\{\alpha_{\vec{Y}}\}_{\vec{Y}}$ of $\mathbb{H}(i)'$, where $\vec{Y} = (Y^1, \dots, Y^k)$ is a fixed point such that $Y^j = 0$ for every $j \in \{1, \dots, k\}$, $j \neq i$.

PROPOSITION 7.10. *Let $\vec{Y} = (Y^1, \dots, Y^k)$ be a fixed point such that $Y^j = 0$ for every $j \in \{1, \dots, k\}$, $j \neq i$. Then*

$$\Phi_i(\alpha_{\vec{Y}}) = J_{Y^i}(x; \beta_i^{-1}).$$

Define $\Lambda'_{\vec{\beta}} = \bigotimes_{i=1}^k \Lambda'_{\beta_i}$ endowed with the symmetric bilinear form $\langle p, q \rangle_{\Lambda'_{\vec{\beta}}} := \prod_{i=1}^k \langle p_i, q_i \rangle_{\Lambda'_{\beta_i}}$ on $\Lambda'_{\vec{\beta}}$ for $p = p_1 \otimes \dots \otimes p_k$, $q = q_1 \otimes \dots \otimes q_k$ in $\Lambda'_{\vec{\beta}}$.

For a k -upla of Young diagrams \vec{Y} , define in \mathcal{H} the operators $\mathfrak{p}_{\vec{Y}} = \prod_{i=1}^k \mathfrak{p}_{\lambda_{Y^i}}$. Summing up, we proved the following.

THEOREM 7.11. *There exists a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -linear isomorphism*

$$\Phi := \bigotimes_{i=1}^k \Phi_i: \mathbb{H}' \rightarrow \Lambda'_{\vec{\beta}}$$

preserving bilinear forms, such that

$$\Phi(\mathfrak{p}_{\vec{Y}}|0\rangle_{\mathbb{H}'}) = p_{Y^1} \otimes \dots \otimes p_{Y^k}, \quad \Phi([\alpha_Y]) = J_{Y^1}(x; \beta_1^{-1}) \otimes \dots \otimes J_{Y^k}(x; \beta_k^{-1}).$$

Moreover, via the isomorphism Φ , the operators \mathfrak{p}_m^i act on $\Lambda'_{\vec{\beta}}$ by multiplication for p_m on the i -th factor if $m < 0$, and as the derivation $m\beta_i^{-1} \frac{\partial}{\partial p_m}$ on the i -th factor if $m > 0$. This makes \mathbb{H}' the Fock space for the Heisenberg algebra $\mathcal{H}_{\mathbb{H}'}$.

7.3.2. The lattice Heisenberg algebra of type A_{k-1} . Let us define now

$$(129) \quad \mathfrak{q}_{-m}^i := \mathfrak{p}_{-m}([D_i]_T) \quad \text{and} \quad \mathfrak{q}_m^i := \mathfrak{p}_m([D_i]_T)$$

for $m \in \mathbb{N}_{>0}$ and $i = 1, \dots, k-1$. By Formula (121) the operators \mathfrak{q}_m^i satisfy the following commutation relations

$$[\mathfrak{q}_m^i, \mathfrak{q}_n^j] = m\delta_{m,-n}C_{i,j} \text{ id for } i, j = 1, \dots, k-1, m, n \in \mathbb{Z} \setminus \{0\},$$

where $C = (c_{ij})$ is the Cartan matrix associated to the Dynkin diagram of type A_{k-1} .

Let $\mathbf{L} \subset H_T^*(X_k)'$ be the \mathbb{Z} -lattice generated by the classes $[D_1]_T, \dots, [D_{k-1}]_T$ with the symmetric bilinear form given by the Cartan matrix C . Then the lattice Heisenberg algebra $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \mathbf{L}}$ associated with \mathbf{L} over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, which has generators \mathfrak{q}_m^i for $m \in \mathbb{Z} \setminus \{0\}$ and $i = 1, \dots, k-1$, is isomorphic to the Heisenberg algebra $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \Omega}$ (cf. Example 2.4) of type A_{k-1} over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ (recall that by Remark 4.26, $\Omega \cong \mathbf{L}$).

Let $E := \sum_{i=0}^k a_i [D_i]_T$ with $a_i \in \mathbb{C}(\varepsilon_1, \varepsilon_2)$, $i = 0, \dots, k$, satisfying the relations

$$(130) \quad 2a_j - a_{j-1} - a_{j+1} = 0 \quad \text{for } j = 1, \dots, k-1,$$

$$(131) \quad a_0\varepsilon_2 + a_k\varepsilon_1 \neq 0.$$

The first condition ensures that

$$\langle [D_i]_T, E \rangle = 0 \quad \text{for } i = 1, \dots, k-1,$$

while by the second $\{[D_1]_T, \dots, [D_{k-1}]_T, E\}$ is a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -basis of $H_T^*(X_k)'$. Note that by (130)

$$\begin{aligned} \langle E, E \rangle &= a_0^2\beta_i^{-1} - a_0a_1 - a_ka_{k-1} + a_k^2\beta_k = \\ &= (k-1)(a_0^2 + a_k^2) - \frac{a_k^2\varepsilon_1}{k\varepsilon_2} - \frac{a_0^2\varepsilon_2}{k\varepsilon_1} - a_0a_1 - a_ka_{k-1}. \end{aligned}$$

Let $\mu := \langle E, E \rangle$ and define $\mathfrak{p}_{-m} := \mathfrak{p}_{-m}(E)$ and $\mathfrak{p}_m := \mathfrak{p}_m(E)$ for $m \in \mathbb{N}_{>0}$. The operators \mathfrak{q}_m^i and \mathfrak{p}_m satisfy the following commutation relations

$$\begin{cases} [\mathfrak{q}_m^i, \mathfrak{q}_n^j] = m\delta_{m,-n}C_{i,j} \text{ id} & \text{for } i, j = 1, \dots, k-1, m, n \in \mathbb{Z} \setminus \{0\}, \\ [\mathfrak{q}_m^i, \mathfrak{p}_n] = 0 & \text{for } i = 1, \dots, k-1, m, n \in \mathbb{Z} \setminus \{0\}, \\ [\mathfrak{p}_m, \mathfrak{p}_n] = m\delta_{m,-n}\mu \text{ id} & \text{for } m, n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Let $\mathbf{L}' \subset H_T^*(X_k)'$ be the \mathbb{Z} -lattice generated by the classes $[D_1]_T, \dots, [D_{k-1}]_T, E$. Then the operators \mathfrak{q}_m^i and \mathfrak{p}_n for $m, n \in \mathbb{Z} \setminus \{0\}$ and $1 \leq i \leq k-1$ define the lattice Heisenberg algebra $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \mathbf{L}'}$ associated to \mathbf{L}' over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$. In particular, $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \mathbf{L}'}$ is the sum (identifying the central elements) of, respectively, the Heisenberg algebra $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \Omega}$ of type A_{k-1} over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ and the Heisenberg algebra $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2)}$ over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ generated by \mathfrak{p}_n , for $n \in \mathbb{Z} \setminus \{0\}$.

Since $\{[D_1]_T, \dots, [D_{k-1}]_T, E\}$ is a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ -basis of $H_T^*(X_k)'$, we get $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \mathbf{L}'} \cong \mathcal{H}_{\mathbb{H}'}$. Hence \mathbb{H}' is the Fock space of $\mathcal{H}_{\mathbb{C}(\varepsilon_1, \varepsilon_2), \mathbf{L}'}$.

REMARK 7.12. By Formula (120), $\mathfrak{q}_m^i = -\beta_i \mathfrak{p}_m^i + \mathfrak{p}_m^{i+1}$ and $\mathfrak{p}_m = \sum_{i=1}^k (a_{i-1} - a_i \beta_i) \mathfrak{p}_m^i$. \triangle

7.3.3. Representation of $\mathcal{A}(1, k)$ on \mathbb{W}' . By our previous results, \mathbb{H}' is an irreducible highest weight representation of $\mathcal{H} + \mathcal{H}_\Omega$, the sum (taken identifying the central elements \mathfrak{c}) of the Heisenberg algebra and the Heisenberg algebra of type A_{k-1} over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ generated, respectively, by the operators \mathfrak{p}_m and \mathfrak{q}_m^i for $m \in \mathbb{Z} \setminus \{0\}$ and $i = 1, \dots, k-1$ (cf. Section 7.3).

We apply the Frenkel-Kac construction (cf. Section 2.3) to the representation $\mathcal{H}_\Omega \rightarrow \text{End}(\mathbb{H}')$ to obtain a representation

$$\widehat{\mathfrak{sl}}_k \rightarrow \text{End}(\mathbb{H}' \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\Omega]) .$$

The representation of \mathcal{H}_Ω is a (non-irreducible) highest weight representation, in which the central element \mathfrak{c} acts as the identity. Thus by Theorem 2.16 the representation of $\widehat{\mathfrak{sl}}_k$ is a highest weight representation with the same highest weight vector, and is level 1. Moreover, we can extend the representation of \mathcal{H} from \mathbb{H}' to $\mathbb{H}' \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\Omega]$ just letting it act as the identity on the group algebra of the root lattice.

From this, we obtain a representation

$$(132) \quad \mathcal{A}(1, k) \cong \mathcal{H} + \widehat{\mathfrak{sl}}_k \rightarrow \text{End}(\mathbb{H}' \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\Omega]) = \text{End}(\mathbb{W}') .$$

From Theorem 2.16 and the discussion above, is not difficult to see that this representation is irreducible, highest weight, and level one, hence it is equivalent to the basic representation of $\mathcal{A}(1, k)$. So we have proved the first statement in (1), Theorem 7.5.

7.3.3.1. Whittaker vectors. Consider now the completed total equivariant cohomology

$$\widehat{\mathbb{W}'} := \prod_{\gamma \in \Omega, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n} .$$

We can extend the isomorphism Θ to

$$\prod_{\gamma \in \Omega, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n} \xrightarrow{\sim} \widehat{\mathbb{H}'} \otimes \left(\prod_{\gamma \in \Omega} \mathbb{C}(\varepsilon_1, \varepsilon_2) \cdot \gamma \right) ,$$

where $\widehat{\mathbb{H}'}$ is the completed total equivariant cohomology of the Hilbert schemes of points on X_k , i.e., $\widehat{\mathbb{H}'} := \prod_n H_T^*(\text{Hilb}^n(X_k))$.

Let $|0\rangle_{\mathbb{W}'}$ be the highest weight vector in \mathbb{W}' .

PROPOSITION 7.13. Fix $\vec{\eta} \in \mathbb{C}(\varepsilon_1, \varepsilon_2)^k$. In the completed total equivariant cohomology $\prod_{\gamma \in \Omega, n \in \mathbb{N}} \mathbb{W}'_{\gamma, n}$, the vector

$$G(\vec{\eta}) := \exp \left(\sum_{i=1}^k \eta_i \mathfrak{p}_{-1}^i \right) |0\rangle$$

is a Whittaker vector of type χ , where $\chi: \mathcal{U}(\mathcal{H}^+ + \mathcal{H}_\Omega^+) \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by

$$\begin{aligned} \chi(h_i \otimes z) &= \eta_{i+1} \beta_{i+1}^{-1} - \eta_i \quad \text{and} \quad \chi(h_i \otimes z^m) = 0, \quad m > 1, i = 1, \dots, k-1, \\ \chi(\mathfrak{p}_1) &= \sum_{i=1}^k \eta_i (\beta_i^{-1} a_{i-1} - a_i) \quad \text{and} \quad \chi(\mathfrak{p}_m) = 0, \quad m > 1. \end{aligned}$$

PROOF. Let us denote by $\mathbb{H}(\hat{i})'$ the completed $\mathbb{H}(i)'$, which is isomorphic by (117) to the completed total equivariant cohomology $\mathbb{H}_{U_i}^{\hat{i}} := \prod_n H_T^*(\text{Hilb}^n(U_i))$. Set

$$G(\eta_i) := \exp(\eta_i \mathfrak{p}_{-1}^i) |0\rangle_{\mathbb{H}(i)'} \in \mathbb{H}(\hat{i})'.$$

Then, by using Theorem 7.11 and the (extended) isomorphisms (118) and (116) we can rewrite the vector $G(\vec{\eta})$ as

$$G(\eta_1) \otimes \dots \otimes G(\eta_k) \otimes \sum_{\gamma \in \Omega} \gamma.$$

By Proposition 6.10, $G(\eta_i)$ is a Whittaker vector for the Heisenberg algebra $\mathcal{H}_i = \langle \mathfrak{p}_m^i \rangle$, which acts on $\mathbb{H}(i)'$, with respect to the character given by

$$(133) \quad \chi(\mathfrak{p}_1^i) = \eta_i \beta_i^{-1} \quad \text{and} \quad \chi(\mathfrak{p}_m^i) = 0 \quad \text{for } m > 1.$$

Again by Theorem 7.11, each copy $\mathcal{H}_i = \langle \mathfrak{p}_m^i \rangle$ acts trivially on $\mathbb{H}(j)'$ for $j \neq i$, and it is easy to see that $G(\vec{\eta})$ is a Whittaker vector for the rank k Heisenberg algebra \mathcal{H}^k (see Example 2.3) with respect to the character (133), for $i = 1, \dots, k$. Then by Remark 7.12, $G(\vec{\eta})$ is a Whittaker vector for $\mathcal{A}(1, k)$ with respect to the character $\chi: \mathcal{U}(\mathcal{H}^+ + \mathcal{H}_\Omega^+) \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$ defined, for every $m > 0$, by

$$\begin{aligned} \chi(\mathfrak{q}_m^i) &= \chi(\mathfrak{p}_m^{i+1}) - \beta_i \chi(\mathfrak{p}_m^i) = \delta_{m,1} (\eta_{i+1} \beta_{i+1}^{-1} - \eta_i), \\ \chi(\mathfrak{p}_m) &= \sum_{i=1}^k (a_{i-1} - a_i \beta_i) \chi(\mathfrak{p}_m^i) = \delta_{m,1} \sum_{i=1}^k \eta_i (\beta_i^{-1} a_{i-1} - a_i). \end{aligned}$$

Thus we obtain the statement just remembering that, by the Frenkel-Kac construction, the operators $h_i \otimes z^m$ for $m > 0$ act as \mathfrak{q}_m^i . \square

7.4. $\mathcal{N} = 2$ $U(1)$ gauge theory on X_k

In this section we complete the proof of Theorem 7.5, studying $U(1)$ gauge theories on X_k in the pure and adjoint masses cases. We use the computations in Chapter 5 to write down the instanton part of the deformed partition function in both cases, then we use the characterization of the Whittaker vectors in Proposition 7.13 to show that the Gaiotto state is a Whittaker vector for the representation of $\mathcal{A}(1, k)$ constructed above. Finally we construct a Carlsson-Okounkov type vertex operator which acts on \mathbb{W}' , show that its supertrace coincides with the instanton part of the deformed partition function for $\mathcal{N} = 2^*$ adjoint matter hypermultiplet, and give a realization of this operator in terms of the generators of the Heisenberg algebra $\mathcal{H}^{\oplus k}$. We do not give an explicit realization in terms of the generators of $\mathcal{H} + \mathcal{H}_\Omega$, but since the two algebras are isomorphic, what we do suffices to conclude the proof of Theorem 7.5.

In this section we always omit to write *inst* for the instanton part of the partition functions, as we used to do in Chapter 5, because we always consider instanton parts.

7.4.1. Pure $\mathcal{N} = 2$ gauge theory. We write down the instanton part of the deformed partition function for the pure $\mathcal{N} = 2$ $U(1)$ gauge theory on X_k . This is by definition (see Formula (87))

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}, \vec{\xi}) &:= \sum_{\gamma \in \Omega, n \in \mathbb{N}} \mathbf{q}^{n + \frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} \int_{\mathcal{M}_{\mathcal{X}_k}(\gamma, n)} [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \\ &= \sum_{\gamma \in \Omega, n \in \mathbb{N}} \mathbf{q}^{\frac{1}{2}\gamma C \gamma} (-\mathbf{q})^n \vec{\xi}^{\vec{\gamma}} \langle [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T, [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \rangle_{\mathbb{W}'} , \end{aligned}$$

where $\vec{\xi}^{\vec{\gamma}} = \prod_{i=1}^{k-1} \xi_i^{\gamma_i}$. By Formula (75), we have

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}, \vec{\xi}) &= \\ &\sum_{\gamma \in \Omega} \mathbf{q}^{\frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} \prod_{i=1}^k \sum_{Y_i} (-\mathbf{q})^{|Y_i|} \cdot \prod_{s \in Y_i} \frac{1}{\left((\ell(s) + 1) \varepsilon_1^{(i)} - a(s) \varepsilon_2^{(i)} \right) \left(\ell(s) \varepsilon_1^{(i)} - (a(s) + 1) \varepsilon_2^{(i)} \right)} . \end{aligned}$$

Recall that by formula (76) we have a factorization in terms of the instanton part of the Nekrasov partition function for the pure gauge theory on \mathbb{C}^2

$$\mathcal{Z}_{ALE}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}, \vec{\xi}) = \sum_{\gamma} \mathbf{q}^{\frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} \prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}; \mathbf{q}) .$$

which is also evident looking at the expression of the Nekrasov partition function for the pure gauge theory on \mathbb{C}^2 computed in Section 6.3.2.

By applying Remark 6.12 we obtain

$$\prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}; \mathbf{q}) = \prod_{i=1}^k \exp \left(\frac{\mathbf{q}}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}} \right) = \exp \left(\frac{\mathbf{q}}{k \varepsilon_1 \varepsilon_2} \right) .$$

Thus

$$\mathcal{Z}_{ALE}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}, \vec{\xi}) = \sum_{\gamma \in \Omega} \mathbf{q}^{\frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} \exp \left(\frac{\mathbf{q}}{k \varepsilon_1 \varepsilon_2} \right) .$$

7.4.1.1. Gaiotto state. Following Section 6.3.2.1, we define the *Gaiotto state* G to be the sum, in the completed basic representation $\hat{\mathbb{W}}'$, of all fundamental classes

$$G := \sum_{\gamma \in \Omega, n \in \mathbb{N}} [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T .$$

We also define the $(\mathbf{q}, \vec{\xi})$ -deformed *Gaiotto state* in $\hat{\mathbb{W}}'_{\mathbf{q}, \vec{\xi}} := \bigoplus_{\gamma \in \Omega, n \in \mathbb{N}} \mathbf{q}^n \vec{\xi}^{\vec{\gamma}} \mathbb{W}'_{\gamma, n}$ as

$$G_{\mathbf{q}, \vec{\xi}} := \sum_{\gamma \in \Omega, n \in \mathbb{N}} \mathbf{q}^{n + \frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T .$$

If we endow $\hat{\mathbb{W}}'_{\mathbf{q}, \vec{\xi}}$ with the scalar product

$$\left\langle \sum_{\gamma, n} \mathbf{q}^{n + \frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} \eta_{\gamma, n}, \sum_{\gamma, n} \mathbf{q}^{n + \frac{1}{2}\gamma C \gamma} \vec{\xi}^{\vec{\gamma}} \nu_{\gamma, n} \right\rangle_{\hat{\mathbb{W}}'_{\mathbf{q}, \vec{\xi}}} := \sum_{\gamma, n} \mathbf{q}^{\frac{1}{2}\gamma C \gamma} (-\mathbf{q})^n \vec{\xi}^{\vec{\gamma}} \langle \eta_{\gamma, n}, \nu_{\gamma, n} \rangle_{\mathbb{W}'_{\gamma, n}} ,$$

it is straightforward that the norm of the $(\mathbf{q}, \vec{\xi})$ -deformed Gaiotto state is the instanton part of the deformed partition function for the $\mathcal{N} = 2$ $U(1)$ gauge theory on X_k :

$$(134) \quad \langle G_{\mathbf{q}, \vec{\xi}}, G_{\mathbf{q}, \vec{\xi}} \rangle_{\mathbb{W}'_{\mathbf{q}, \vec{\xi}}} = \mathcal{Z}_{ALE}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2; \mathbf{q}, \vec{\xi}).$$

Now we can prove the second part of statement (1) in Theorem 7.5.

THEOREM 7.14. *The Gaiotto state is a Whittaker vector for the action of $\mathcal{A}(1, k)$ on \mathbb{W}' , of type $\chi: \mathcal{U}(\mathcal{H}^+ + \mathcal{H}_{\Omega}^+) \rightarrow \mathbb{C}(\varepsilon_1, \varepsilon_2)$, where χ is defined by*

$$\begin{aligned} \chi(h_i \otimes z^m) &= 0 \quad i = 1, \dots, k-1, m > 0, \\ \chi(\mathbf{p}_m) &= \delta_{m,1} \sum_{i=1}^k \sqrt{\frac{\beta_i}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}}} (\beta_i^{-1} a_{i-1} - a_i) \quad m > 0. \end{aligned}$$

PROOF. Under the isomorphisms (116) and (118), the Gaiotto state becomes

$$G = \bigotimes_{i=1}^k \sum_n [\text{Hilb}^n(U_i)]_T \otimes \sum_{\gamma \in \mathbb{Q}} \gamma \in \bigotimes_{i=1}^k \mathbb{H}_{U_i}^{\hat{\gamma}} \otimes \prod_{\gamma \in \Omega} \mathbb{C}(\varepsilon_1, \varepsilon_2) \gamma.$$

By Proposition 6.13, $\sum_n [\text{Hilb}^n(U_i)]_T$ is the Whittaker vector $G(\gamma_i)$ for the Heisenberg algebra \mathcal{H}_i with $\gamma_i = \sqrt{\frac{\beta_i}{\varepsilon_1^{(i)} \varepsilon_2^{(i)}}}$. It follows that $G = G(\vec{\gamma})$ as in Proposition 7.13, with $\vec{\gamma} = (\gamma_1, \dots, \gamma_k)$ defined above, is a Whittaker vector for $\mathcal{H} \oplus \hat{\mathfrak{sl}}_k$ of type

$$\begin{aligned} \chi(h_i \otimes z^m) &= \delta_{m,1} (\gamma_{i+1} \beta_{i+1}^{-1} - \gamma_i), \\ \chi(\mathbf{p}_m) &= \delta_{m,1} \sum_{i=1}^k \gamma_i (\beta_i^{-1} a_{i-1} - a_i). \end{aligned}$$

□

7.4.2. $\mathcal{N} = 2^*$ $U(1)$ gauge theory. Recall that the instanton part of the deformed partition function with one adjoint hypermultiplet of mass m is (see Formula (94))

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}, \vec{\xi}) &:= \sum_{\gamma \in \Omega, n \in \mathbb{N}} \mathbf{q}^{n+\frac{1}{2}\gamma C \gamma \vec{\xi} \vec{\gamma}} \int_{\mathcal{M}_{\mathcal{X}_k}(\gamma, n)} [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \\ &= \sum_{\gamma \in \Omega, n \in \mathbb{N}} \mathbf{q}^{\frac{1}{2}\gamma C \gamma} (-\mathbf{q})^n \vec{\xi} \vec{\gamma} \langle [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T, [\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \rangle_{\mathbb{W}' \otimes H_{T_m}^*(\text{pt})}. \end{aligned}$$

where we have identified the fundamental class $[\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T$ with the class in $\mathbb{W}' \otimes H_{T_m}^*(\text{pt})$ given by $[\mathcal{M}_{\mathcal{X}_k}(\gamma, n)]_T \otimes 1$, and we are using the following extension of the scalar product on \mathbb{W}' :

$$\langle c \otimes p, d \otimes q \rangle_{\mathbb{W}' \otimes H_{T_m}^*(\text{pt})} \mapsto \langle c, d \rangle_{\mathbb{W}'} p \cdot q \in \mathbb{C}(\varepsilon_1, \varepsilon_2)[m].$$

By Formula (90), we have

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}, \vec{\xi}) &= \\ &= \sum_{\gamma \in \Omega} \mathbf{q}^{\frac{1}{2}\gamma C \gamma \vec{\xi} \vec{\gamma}} \prod_{i=1}^k \sum_{Y^i} \mathbf{q}^{|Y^i|} \prod_{s \in Y^i} \frac{(\ell(s) + 1)\varepsilon_1^{(i)} - a(s)\varepsilon_2^{(i)} + m}{(\ell(s) + 1)\varepsilon_1^{(i)} - a(s)\varepsilon_2^{(i)}} \cdot \frac{\ell(s)\varepsilon_1^{(i)} - (a(s) + 1)\varepsilon_2^{(i)} - m}{\ell(s)\varepsilon_1^{(i)} - (a(s) + 1)\varepsilon_2^{(i)}}. \end{aligned}$$

By Formula (91) we have also the factorization

$$\mathcal{Z}_{ALE}^{\mathcal{N}=2*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}, \vec{\xi}) = \sum_{\gamma} \mathbf{q}^{\frac{1}{2}\gamma C\gamma} \vec{\xi}^{\gamma} \prod_{i=1}^k \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, m; \mathbf{q}) .$$

By using Proposition 6.15, we obtain

$$\begin{aligned} \mathcal{Z}_{ALE}^{\mathcal{N}=2*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}, \vec{\xi}) &= \sum_{\gamma \in \Omega} \mathbf{q}^{\frac{1}{2}\gamma C\gamma} \vec{\xi}^{\gamma} \prod_{i=1}^k \prod_{l=1}^{\infty} (1 - \mathbf{q}^l)^{m(m - \varepsilon_1^{(i)} - \varepsilon_2^{(i)}) - 1} \\ (135) \qquad \qquad \qquad &= \sum_{\gamma \in \Omega} \mathbf{q}^{\frac{1}{2}\gamma C\gamma} \vec{\xi}^{\gamma} \prod_{l=1}^{\infty} (1 - \mathbf{q}^l)^{k(m(m - \varepsilon_1 - \varepsilon_2) - 1)} . \end{aligned}$$

7.4.2.1. A Carlsson-Okounkov type operator. First note that one can define Carlsson-Okounkov type operators on \mathbb{H}' depending on line bundles on X_k in a way similar to that described in Section 6.3.3.1 for \mathbb{C}^2 . Let us consider the Carlsson-Okounkov type operator $W_k(\mathcal{O}_{X_k}(m), z) \in \text{End}(\mathbb{H}')[[z, z^{-1}]]$ depending on the trivial line bundle $\mathcal{O}_{X_k}(m)$ on X_k with an action of T_m given by scaling the fibers. By the isomorphism (116) and the relation between universal objects as described in Section 5.1.2, one can interpret $W_k(\mathcal{O}_{X_k}(m), z)$ as an operator in $\text{End}(\mathbb{W}')[[z, z^{-1}]]$.

Define the operators \mathbf{q} and $\vec{\xi}$ on \mathbb{W}' such that

$$\mathbf{q}^{\mathcal{N}}|_{\mathbb{W}'_{\gamma,n}} := \mathbf{q}^{n+\frac{1}{2}\gamma C\gamma} \text{id} \quad \text{and} \quad \vec{\xi}^{\gamma}|_{\mathbb{W}'_{\gamma,n}} := \prod_{i=1}^{k-1} \xi_i^{\gamma_i} \text{id} .$$

The supertrace of the operator $\mathbf{q}^{\mathcal{N}} \vec{\xi}^{\gamma} W_k(\mathcal{O}_{X_k}(m), z)$ is

$$\begin{aligned} \text{str } \mathbf{q}^{\mathcal{N}} \vec{\xi}^{\gamma} W_k(\mathcal{O}_{X_k}(m), z) &= \sum_{\gamma, n} \text{str} \left(\mathbf{q}^{n+\frac{1}{2}\gamma C\gamma} \vec{\xi}^{\gamma} W_k(\mathcal{O}_{X_k}(m), z) \right) |_{\mathbb{W}'_{\gamma,n}} \\ &= \sum_{\gamma \in \Omega} \mathbf{q}^{\frac{1}{2}\gamma C\gamma} \vec{\xi}^{\gamma} \sum_{n \geq 0} \text{str} \left(\mathbf{q}^{\mathcal{N}} W_k(\mathcal{O}_{X_k}(m), z) \right) |_{\mathbb{H}'_n} . \end{aligned}$$

By using the factorization property of $W_k(\mathcal{O}_{X_k}(m), z)$ with respect to the isomorphism (118) (cf.[28, Section 3.1]), we get

$$(136) \qquad \qquad \qquad W_k(\mathcal{O}_{X_k}(m), z) = \bigotimes_{i=1}^k W_i(\mathcal{O}_{U_i}(m), z),$$

where $\mathcal{O}_{U_i}(m)$ is the trivial line bundle on U_i with an action of T_m which rescales the fibers. Therefore

$$\begin{aligned} \sum_{n \geq 0} \text{str} \left(\mathbf{q}^{\mathcal{N}} W_k(\mathcal{O}_{X_k}(m), z) \right) |_{\mathbb{H}'_n} &= \prod_{i=1}^k \sum_{n_i \geq 0} \text{str} \left(\mathbf{q}^{n_i} W_i(\mathcal{O}_{U_i}(m), z) \right) |_{H_T^*(\text{Hilb}^{n_i}(U_i))'} \\ &= \prod_{i=1, \dots, k} \mathcal{Z}_{\mathbb{R}^4}^{\mathcal{N}=2*}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, m; \mathbf{q}) , \end{aligned}$$

where the last line follows from Formula (111). Thus by Formula (135), we obtain

$$\text{str } \mathbf{q}^{n+\frac{1}{2}\gamma C\gamma} \vec{\xi}^{\gamma} W_k(\mathcal{O}_{X_k}(m), z) = \mathcal{Z}_{ALE}^{\mathcal{N}=2*}(\varepsilon_1, \varepsilon_2, m; \mathbf{q}, \vec{\xi}) .$$

REMARK 7.15. By using the factorization property (136) and Formula (112), we get an expression of $W_k(\mathcal{O}_{X_k}(m), z)$ depending on operators \mathfrak{p}_m^j for $m \in \mathbb{Z} \setminus \{0\}$:

$$W_k(\mathcal{O}_{X_k}(m), z) = \exp \left(\sum_{i>0} \frac{(-1)^i z^i}{i} \sum_{j=1}^k \frac{m}{\varepsilon_2^{(i)}} \mathfrak{p}_{-i}^j \right) \exp \left(\sum_{i>0} \frac{(-1)^i z^{-i}}{i} \sum_{j=1}^k \frac{\varepsilon_1^{(i)} + \varepsilon_2^{(i)} - m}{\varepsilon_2^{(i)}} \mathfrak{p}_i^j \right).$$

Therefore $W_k(\mathcal{O}_{X_k}(m), z)$ is a vertex operator depending only on the Cartan subalgebra $\mathcal{H}^k \cong \mathcal{H} \oplus \mathcal{H}_\Omega \subset \mathcal{A}(1, k)$. So we have proved part (2) of Theorem 7.5. \triangle

APPENDIX A

Serre duality for smooth projective stacks

In this section we give a proof of Serre duality theorems for coherent sheaves on smooth projective stacks. These results are only sketched in Nironi's papers [91, 90], thus we follow here the more complete treatment in [23]. First, we recall two results from [90, Theorem 1.16 and Corollary 2.10]. Then we prove Serre duality for Deligne-Mumford stacks.

PROPOSITION A.1. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of separated Deligne-Mumford stacks of finite type over k . The functor $\mathbf{R}f_*: \mathbf{D}^+(\mathcal{X}) \rightarrow \mathbf{D}^+(\mathcal{Y})$ has a right adjoint $f^!: \mathbf{D}^+(\mathcal{Y}) \rightarrow \mathbf{D}^+(\mathcal{X})$. Moreover, for $\mathcal{E}^\bullet \in \mathbf{D}_c^+(\mathcal{X})$ and $\mathcal{F}^\bullet \in \mathbf{D}^+(\mathcal{Y})$ the natural morphism*

$$(137) \quad \mathbf{R}f_* \mathbf{R}\mathcal{H}om_{\mathcal{X}}(\mathcal{E}^\bullet, f^! \mathcal{F}^\bullet) \rightarrow \mathbf{R}\mathcal{H}om_{\mathcal{Y}}(\mathbf{R}f_* \mathcal{E}^\bullet, \mathbf{R}f_* f^! \mathcal{F}^\bullet) \xrightarrow{\mathrm{tr}_f} \mathbf{R}\mathcal{H}om_{\mathcal{Y}}(\mathbf{R}f_* \mathcal{E}^\bullet, \mathcal{F}^\bullet)$$

is an isomorphism.

THEOREM A.2. *(Serre duality - I) Let $p: \mathcal{X} \rightarrow \mathrm{Spec}(k)$ be a proper Cohen-Macaulay Deligne-Mumford stack of pure dimension d . For any coherent sheaf \mathcal{E} on \mathcal{X} one has*

$$H^i(\mathcal{X}, \mathcal{E})^\vee \simeq \mathrm{Ext}^{d-i}(\mathcal{E}, \omega_{\mathcal{X}}),$$

where $\omega_{\mathcal{X}}$ is the dualizing sheaf of \mathcal{X} .

PROOF. By [90, Corollary 2.30], $p^! \mathcal{O}_{\mathrm{Spec}(k)}$ is isomorphic to the complex $\omega_{\mathcal{X}}[d]$, where $\omega_{\mathcal{X}}$ is the dualizing sheaf of \mathcal{X} . Let \mathcal{E} be a coherent sheaf on \mathcal{X} . By applying the Formula (137) to the coherent sheaves \mathcal{E} and $\mathcal{O}_{\mathrm{Spec}(k)}$ (regarded in the derived category as complexes concentrated in degree zero), we obtain

$$\mathbf{R}p_* \mathbf{R}\mathcal{H}om_{\mathcal{X}}(\mathcal{E}, \omega_{\mathcal{X}}[d]) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathrm{Spec}(k)}(\mathbf{R}p_* \mathcal{E}, \mathcal{O}_{\mathrm{Spec}(k)}) \simeq \mathbf{R}\Gamma(\mathcal{X}, \mathcal{E})^\vee.$$

By taking cohomology, we get for any $i \geq 0$

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{E}, \omega_{\mathcal{X}}[d-i]) \simeq H^i(\mathcal{X}, \mathcal{E})^\vee.$$

Since the category of quasi-coherent sheaves on \mathcal{X} has enough injectives ([90, Proposition 1.13]), we get $\mathrm{Ext}^{d-i}(\mathcal{E}, \omega_{\mathcal{X}}) \simeq \mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{E}, \omega_{\mathcal{X}}[d-i])$, and therefore we obtain the desired result. \square

Now we would like to prove a Serre duality theorem for Ext groups. We readapt the proof of the analogous theorem in the case of coherent sheaves on proper Gorenstein varieties (cf. [11, Appendix C]). From now on, we assume that \mathcal{X} is a smooth projective stack of dimension d , so that it is of the form $[Z/G]$ with Z a smooth quasi-projective variety (cf. Remark 1.17). Recall that any G -equivariant coherent sheaf on Z admits a finite resolution consisting of G -equivariant locally free sheaves of finite rank ([29, Proposition 5.1.28]). Then we get the following result.

LEMMA A.3. [23, Lemma B.3] *A coherent sheaf on \mathcal{X} admits a finite resolution by locally free sheaves of finite rank.*

Before proving Serre duality theorem for Ext group we need some technical results about the relation between the derived dual $(\cdot)^*$ of a coherent sheaf and the tensor product $\otimes^{\mathbf{L}}$ in the derived category of \mathcal{X} . The techniques we shall use are similar to those in the proofs of [11, Proposition A.86, Proposition A.87 and Corollary A.88].

LEMMA A.4. [23, Lemma B.4] *Let \mathcal{E} , \mathcal{F} and \mathcal{G} be coherent sheaves on \mathcal{X} . There is a functorial isomorphism*

$$\mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbf{L}} \mathcal{G} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G})$$

in the derived category.

PROOF. Let $\mathcal{E}^{\bullet} \rightarrow \mathcal{E}$ and $\mathcal{G}^{\bullet} \rightarrow \mathcal{G}$ be finite resolutions of \mathcal{E} and \mathcal{G} , respectively, consisting of locally free sheaves of finite rank. There is a quasi-isomorphism of complexes

$$(138) \quad \mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}^{\bullet}, \mathcal{F}) \otimes \mathcal{G}^{\bullet} \simeq \mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}^{\bullet}, \mathcal{F} \otimes \mathcal{G}^{\bullet}).$$

Let $\mathcal{F} \rightarrow \mathcal{F}^{\bullet}$ be an injective resolution of \mathcal{F} . Then $\mathcal{J}^{\bullet} = \mathcal{F}^{\bullet} \otimes \mathcal{G}^{\bullet}$ is injective and quasi-isomorphic to $\mathcal{F} \otimes \mathcal{G}^{\bullet}$. There is an induced quasi-isomorphism

$$\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}^{\bullet}, \mathcal{F}^{\bullet}) \otimes \mathcal{G}^{\bullet} \rightarrow \mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}^{\bullet}, \mathcal{J}^{\bullet}),$$

which yields in derived category the required isomorphism. \square

LEMMA A.5. [23, Lemma B.5] *Let \mathcal{E} , \mathcal{F} be coherent sheaves on \mathcal{X} and \mathcal{M}^{\bullet} a finite complex of locally free sheaves of finite rank. Then*

$$(139) \quad \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{M}^{\bullet}, \mathcal{F}) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{M}^{\bullet}, \mathcal{F})).$$

PROOF. Let \mathcal{I}^{\bullet} be an injective resolution of \mathcal{F} . There is an isomorphism of complexes

$$(140) \quad \mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E} \otimes \mathcal{M}^{\bullet}, \mathcal{I}^{\bullet}) \simeq \mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{M}^{\bullet}, \mathcal{I}^{\bullet})).$$

The left-hand side produces in derived category the object $\mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{M}^{\bullet}, \mathcal{F})$. To deal with the right-hand side, we note that since \mathcal{M}^{\bullet} is flat and \mathcal{I}^{\bullet} is injective, the complex $\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{M}^{\bullet}, \mathcal{I}^{\bullet})$ is injective (and is quasi-isomorphic to $\mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{M}^{\bullet}, \mathcal{F})$). Therefore the right-hand side of eq. (140) in derived category yields $\mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{M}^{\bullet}, \mathcal{F}))$. \square

PROPOSITION A.6. [23, Proposition B.6] *Let \mathcal{E} , \mathcal{F} and \mathcal{G} coherent sheaves on \mathcal{X} . Then in the derived category of \mathcal{X} there are functorial isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{G}^{*\bullet}, \mathcal{F}) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E}, \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}), \\ \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{G}, \mathcal{F}) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E}, \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}^{*\bullet}), \end{aligned}$$

where $\mathcal{G}^{*\bullet}$ denotes the derived dual $\mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{G}, \mathcal{O}_{\mathcal{X}})$ of \mathcal{G} .

PROOF. Since \mathcal{G} admits a finite resolution consisting of finite rank locally free sheaves, its derived dual $\mathcal{G}^{*\bullet}$ is isomorphic, in the derived category of \mathcal{X} , to a finite complex consisting of finite rank locally free sheaves. By Lemma (A.4), we get

$$\mathcal{G}^{*\bullet} \otimes^{\mathbf{L}} \mathcal{F} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{X}}(\mathcal{G}, \mathcal{F}) .$$

By eq. (139), we have

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{G}^{*\bullet}, \mathcal{F}) &\simeq \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{G}^{*\bullet}, \mathcal{F})) \\ &\simeq \mathbf{R}\mathcal{H}om_{\mathcal{X}}^{\bullet}(\mathcal{E}, \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) . \end{aligned}$$

By taking cohomology, we obtain

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{G}^{*\bullet}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E}, \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) .$$

Similarly, we get the second formula of the statement. \square

THEOREM A.7 (Serre duality - II). [23, Theorem B.7] *Let $p: \mathcal{X} \rightarrow \mathrm{Spec}(k)$ be a smooth projective stack of pure dimension d . Let \mathcal{E} and \mathcal{F} be coherent sheaves on \mathcal{X} . Then*

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{G}) \simeq \mathrm{Ext}^{d-i}(\mathcal{G}, \mathcal{E} \otimes \omega_{\mathcal{X}})^{\vee} ,$$

where $\omega_{\mathcal{X}}$ is the canonical line bundle of \mathcal{X} .

PROOF. By [90], Theorem 2.22, the dualizing sheaf of \mathcal{X} is the canonical line bundle $\omega_{\mathcal{X}}$. By applying the Formula (137) to the complexes $\mathcal{E}^{*\bullet} \otimes^{\mathbf{L}} \mathcal{G}$ and $\mathcal{O}_{\mathrm{Spec}(k)}$ we get

$$\mathbf{R}p_* \mathbf{R}\mathcal{H}om_{\mathcal{X}}(\mathcal{E}^{*\bullet} \otimes^{\mathbf{L}} \mathcal{G}, \omega_{\mathcal{X}}[d]) \simeq \mathbf{R}\Gamma(\mathcal{X}, \mathcal{E}^{*\bullet} \otimes^{\mathbf{L}} \mathcal{G})^{\vee} .$$

By taking cohomology and by Proposition A.6, we obtain

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E}^{*\bullet} \otimes^{\mathbf{L}} \mathcal{G}, \omega_{\mathcal{X}}[d]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{G}, \mathcal{E} \otimes^{\mathbf{L}} \omega_{\mathcal{X}}[d])$$

in the left-hand-side, and

$$H^0(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{E}^{*\bullet} \otimes^{\mathbf{L}} \mathcal{G}))^{\vee} \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E}, \mathcal{G})^{\vee}$$

in the right-hand side. Therefore

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{E}, \mathcal{G}[i]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathcal{G}, \mathcal{E} \otimes^{\mathbf{L}} \omega_{\mathcal{X}}[d-i])^{\vee} \simeq \mathrm{Ext}^{d-i}(\mathcal{G}, \mathcal{E} \otimes \omega_{\mathcal{X}})^{\vee} .$$

\square

APPENDIX B

Töen-Riemann-Roch theorem

Here we briefly survey the Töen-Riemann-Roch theorem (cf [102, 103]), which is an analog for stacks of the Riemann-Roch theorem. A basic feature is that the integral of the product of the Chern character and the Todd class of the stack is not computed over \mathcal{X} but over its inertia stack $\mathcal{I}(\mathcal{X})$. So one needs to send, in a suitable way, the elements in the K-theory of \mathcal{X} to $H_{rep}^\bullet(\mathcal{X})$, which is by definition the étale cohomology $H_{et}^\bullet(\mathcal{I}(\mathcal{X}))$ of its inertia stack, and then one performs the usual integral. We follow the presentation of [19, Appendix C].

Let \mathcal{X} be a smooth proper Deligne-Mumford stack, and $\mathcal{I}(\mathcal{X})$ its inertia stack. One can define a local immersion $\pi: \mathcal{I}(\mathcal{X}) \rightarrow \mathcal{X}$ such that for any scheme S the morphism $\pi(S)$ sends $(x, g) \in \mathcal{I}(\mathcal{X})(S)$ to $x \in \mathcal{X}(S)$.

Let \mathcal{F} be a locally free sheaf on $\mathcal{I}(\mathcal{X})$. There is a canonical automorphism of \mathcal{F} induced by the 2-morphism $\pi \Rightarrow \pi$ such that $(x, g) \mapsto g$. This gives a decomposition

$$\mathcal{F} = \bigoplus_{\omega \in \mu_\infty} \mathcal{F}^\omega ,$$

where μ_∞ is the set of all roots of unity in \mathbb{C} , and the canonical automorphism acts by multiplication by ω on each \mathcal{F}^ω . Thus we can define an endomorphism ρ of $K(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C}$ by

$$\rho(\mathcal{F}) = \sum_{\omega} \omega [\mathcal{F}^\omega] .$$

By [103, Section 3.3], there is a canonical morphism $can: K(\mathcal{I}(\mathcal{X})) \rightarrow K_{et}(\mathcal{I}(\mathcal{X}))$ into the étale K-theory of the inertia stack. We define the *Frobenius character* for locally free sheaves on \mathcal{X} as the composition

$$\phi: K(\mathcal{X}) \otimes \mathbb{C} \xrightarrow{\pi^*} K(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C} \xrightarrow{\rho} K(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C} \xrightarrow{can} K_{et}(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C} .$$

To define an analog of the Todd character we need the following construction. Let N be the normal bundle to the local immersion $\pi: \mathcal{I}(\mathcal{X}) \rightarrow \mathcal{X}$, and define

$$\alpha := can \circ \rho(\lambda_{-1}(N^\vee)) \in K_{et}(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C} ,$$

where $\lambda_{-1}(N^\vee) = \sum_{i \geq 0} (-1)^i \Lambda^i N^\vee \in K(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C}$. It can be shown that α is an invertible element in $K_{et}(\mathcal{I}(\mathcal{X})) \otimes \mathbb{C}$.

Define the *cohomology with coefficients in the representations* $H_{rep}^\bullet(\mathcal{X}) := H_{et}^\bullet(\mathcal{I}(\mathcal{X}))$. We define the *Chern character* $ch^{rep}: K(\mathcal{X}) \rightarrow H_{rep}^\bullet(\mathcal{X}) \otimes \mathbb{C}$ by

$$ch^{rep}(\mathcal{F}) = ch^{et}(\phi(\mathcal{F})) ,$$

where $\text{ch}^{et}: K_{et}(\mathcal{I}(\mathcal{X})) \rightarrow H_{et}^\bullet(\mathcal{I}(\mathcal{X})) = H_{rep}^\bullet(\mathcal{X})$ is the usual Chern character for the inertia stack. The Todd class of \mathcal{X} is defined as

$$\text{Td}^{rep}(\mathcal{X}) = \text{ch}^{et}(\alpha^{-1}) \text{Td}^{et}(\mathcal{T}_{\mathcal{I}(\mathcal{X})}) ,$$

where $\mathcal{T}_{\mathcal{I}(\mathcal{X})}$ is the tangent sheaf to the inertia stack. Now we can state the Töen-Riemann-Roch theorem.

THEOREM B.1. *let \mathcal{X} be a smooth proper Deligne-Mumford stack, \mathcal{F} a locally free sheaf on \mathcal{X} . Then*

$$\chi(\mathcal{X}, \mathcal{F}) = \int_{\mathcal{I}(\mathcal{X})}^{rep} \text{ch}^{rep}(\mathcal{F}) \text{Td}^{rep}(\mathcal{X}) ,$$

where $\int_{\mathcal{X}}^{rep}$ is the push-forward with respect to $p: \mathcal{X} \rightarrow \text{Spec } \mathbb{C}$.

Note that the unity section of the inertia stack $1: \mathcal{X} \rightarrow \mathcal{I}(\mathcal{X})$ induces a decomposition

$$H_{rep}^\bullet(\mathcal{X}) = H_{et}^\bullet(\mathcal{I}(\mathcal{X})) \cong H_{et}^\bullet(\mathcal{X}) \oplus H_{et}^\bullet(\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}) .$$

Given a class $x \in H_{rep}^\bullet(\mathcal{X})$, denote by $x = x_1 + x_{\neq 1}$ the corresponding decomposition. Thus one has

$$\int_{\mathcal{X}}^{rep} x = \int_{\mathcal{I}(\mathcal{X})}^{et} x = \int_{\mathcal{X}}^{et} x_1 + \int_{\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}}^{et} x_{\neq 1} .$$

Moreover, one can show that $\text{ch}_1^{rep} = \text{ch}^{et}$ and $\text{Td}_1^{rep} = \text{Td}^{et}$ (cf. [19, Lemmas C.2 and C.3]), so that one can restate the theorem in the following form:

$$(141) \quad \chi(\mathcal{X}, \mathcal{F}) = \int_{\mathcal{X}}^{et} \text{ch}^{et}(\mathcal{F}) \text{Td}^{et}(\mathcal{X}) + \int_{\mathcal{I}(\mathcal{X}) \setminus \mathcal{X}}^{rep} \text{ch}^{rep}(\mathcal{F}) \text{Td}^{rep}(\mathcal{X}) .$$

APPENDIX C

The dimension of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$

In this appendix the dimension of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is computed explicitly by using the Töen-Riemann-Roch theorem (see Appendix B). In particular we prove the following result.

THEOREM C.1. *Let $s \in \mathbb{Z}$, $\vec{w} \in \mathbb{N}^k$ and $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ be the moduli space of $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ -framed sheaves on \mathcal{X}_k of rank r , first Chern class $\sum_{i=1}^{k-1} u_i \omega_i$ and determinant Δ . Then*

$$\dim_{\mathbb{C}} \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) = 2r\Delta - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})_{j,j} \vec{w}(0) \cdot \vec{w}(j) .$$

We need to introduce some preliminaries. Before doing the computation in Section 3.3, we study the inertia stack of \mathcal{X}_k in Section 3.1, and compute in Section 3.2 some topological invariants of \mathcal{X}_k and \mathcal{D}_∞ .

3.1. The inertia stack of \mathcal{X}_k

In this section we will compute the inertia stack $\mathcal{I}(\mathcal{X}_k)$ of \mathcal{X}_k . As we saw in Appendix B, this is a fundamental ingredient for applying the Töen-Riemann-Roch theorem.

3.1.1. Characterization of the stacky points p_0 and p_∞ . Here we give a characterization of the stacky points p_0 and p_∞ of $\mathcal{D}_\infty \subset \mathcal{X}_k$ as trivial gerbes over a point. Moreover, we characterize their Picard groups and the restrictions to them of the generators of the Picard group of \mathcal{D}_∞ .

LEMMA C.2. *Both stacks p_0 and p_∞ are isomorphic to $\mathcal{B}\mu_{k\tilde{k}} = [pt/\mu_{k\tilde{k}}]$. At a gerbe structure level, the maps between the banding groups μ_k of \mathcal{D}_∞ and $\mu_{k\tilde{k}}$ of p_0 and p_∞ are given by*

$$\begin{aligned} \mu_k &\rightarrow \mu_{k\tilde{k}} , \\ \omega &\mapsto \omega^{\pm\tilde{k}} , \end{aligned}$$

where we take the minus sign for p_0 , and plus for p_∞ .

PROOF. First consider $\sigma_{\infty,k+2}$. Using Section 1.6.2 we can compute the quotient stacky fan $\bar{\Sigma}_k/\sigma_{\infty,k+2}$. First note that $N(\sigma_{\infty,k+2}) \simeq \mathbb{Z}^2/(\mathbb{Z}v_0 + k\mathbb{Z}v_\infty) \simeq \mathbb{Z}_{k\tilde{k}}$, and the quotient map $N \rightarrow N(\sigma_{\infty,k+2})$ sends $ae_1 + be_2$ to $a \bmod k\tilde{k}$. The quotient fan $\bar{\Sigma}_k/\sigma_{\infty,k+2} \subset N(\sigma_{\infty,k+2})_{\mathbb{Q}} = 0$ is just $\{0\}$. Thus p_0 is the $\mu_{k\tilde{k}}$ -trivial gerbe $\mathcal{B}\mu_{k\tilde{k}} := [pt/\mu_{k\tilde{k}}]$ over the point pt .

The quotient map $N \rightarrow N(\sigma_{\infty,k+2})$ factorizes through the quotient map $N(\rho_\infty) \rightarrow N(\sigma_{\infty,k+2})$, which is

$$(c, d) \mapsto \begin{cases} c - d\tilde{k} \bmod k\tilde{k} & \text{for } k \text{ even} , \\ c\frac{k-1}{2} - d\tilde{k} \bmod k\tilde{k} & \text{for } k \text{ odd} . \end{cases}$$

The induced map between the torsion subgroups $\mathbb{Z}_k \rightarrow \mathbb{Z}_{k\tilde{k}}$ is the multiplication by $-\tilde{k}$, and the map between the banding groups of \mathcal{D}_∞ and p_0 is given by

$$\mu_k \simeq \text{Hom}(N(\rho_\infty)^{tor}, \mathbb{C}^*) \xrightarrow{(\cdot)^{-\tilde{k}}} \text{Hom}(N(\sigma_{\infty,k+2})^{tor}, \mathbb{C}^*) \simeq \mu_{k\tilde{k}} .$$

For p_∞ one does the same. \square

Now we give a characterization of the line bundles over p_0 and p_∞ , seen as $\mu_{k\tilde{k}}$ -trivial gerbes over a point.

LEMMA C.3. *The Picard group $\text{Pic}(p_0)$ (resp. $\text{Pic}(p_\infty)$) of p_0 (resp. p_∞) is generated by the line bundle \mathcal{L}_{p_0} (resp. \mathcal{L}_{p_∞}) corresponding to the character $\chi: \omega \in \mu_{k\tilde{k}} \rightarrow \omega \in \mathbb{C}^*$. In particular, $\text{Pic}(p_0) \simeq \text{Pic}(p_\infty) \simeq \mathbb{Z}_{k\tilde{k}}$. The restrictions of the generators $\mathcal{L}_1, \mathcal{L}_2$ of $\text{Pic}(\mathcal{D}_\infty)$ to p_0, p_∞ behave as follows.*

$$\begin{aligned} \mathcal{L}_1|_{p_0} &\simeq \mathcal{L}_{p_0} & \mathcal{L}_1|_{p_\infty} &\simeq \mathcal{L}_{p_\infty} , \\ \mathcal{L}_2|_{p_0} &\simeq \begin{cases} \mathcal{L}_{p_0}^{\otimes \tilde{k}} & \text{for } k \text{ even}, \\ \mathcal{L}_{p_0}^{\otimes 2k} & \text{for } k \text{ odd}; \end{cases} & \mathcal{L}_2|_{p_\infty} &\simeq \begin{cases} \mathcal{L}_{p_\infty}^{\otimes -\tilde{k}} & \text{for } k \text{ even}, \\ \mathcal{L}_{p_\infty}^{\otimes -2k} & \text{for } k \text{ odd}. \end{cases} \end{aligned}$$

PROOF. First consider $\sigma_{\infty,k+2}$. By arguing as in the proof of Lemma 4.30, one sees that the restrictions of the line bundles on \mathcal{X}_k behave as follows:

$$\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0)|_{p_0} \simeq \begin{cases} \mathcal{L}_{p_0}^{\otimes k} & \text{for } k \text{ even} , \\ \mathcal{L}_{p_0}^{\otimes 2k} & \text{for } k \text{ odd} . \end{cases} , \quad \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{p_0} \simeq \mathcal{L}_{p_0} .$$

So, we obtain

$$\mathcal{L}_1|_{p_0} \simeq \mathcal{L}_{p_0} , \quad \mathcal{L}_2|_{p_0} \simeq \begin{cases} \mathcal{L}_{p_0}^{\otimes \tilde{k}} & \text{for } k \text{ even} , \\ \mathcal{L}_{p_0}^{\otimes 2k} & \text{for } k \text{ odd} . \end{cases}$$

For p_∞ one does the same. \square

3.1.2. Characterization of the inertia stack $\mathcal{I}(\mathcal{X}_k)$. Recall that, by Theorem 1.69, given a 2-dimensional toric Deligne-Mumford stack $\mathcal{X} = [Z_\Sigma/G_\Sigma]$ with stacky fan $\Sigma = (N, \Sigma, \beta)$ with Σ is complete, its inertia stack has a description depending on the *boxes* of the maximal cones of its stacky fan. For each 2-dimensional cone $\sigma \in \Sigma(2)$ consider

$$\text{Box}(\sigma) = \{v \in N \mid \bar{v} = \sum_{\rho_i \subset \sigma} q_i \bar{v}_i \text{ for some } 0 \leq q_i < 1\} ,$$

and let $\text{Box}(\Sigma)$ be the union of $\text{Box}(\sigma)$ for all 2-dimensional cones $\sigma \in \Sigma$. For each $v \in N$, we denote by $\sigma(\bar{v})$ the unique minimal cone containing \bar{v} . By Theorem 1.69 there is a one-to-one correspondence between elements $v \in \text{Box}(\Sigma)$ and elements in $g \in G_\Sigma$ which fix a point of

Z_Σ . The closed substack corresponding to the cone $\sigma(\bar{v})$ has a quotient stack description as $\mathcal{X}(\Sigma/\sigma(\bar{v})) \simeq [Z_\Sigma^g/G_\Sigma]$. Moreover, we get

$$\mathcal{I}(\mathcal{X}) = \bigsqcup_{g \in G} [Z_\Sigma^g/G_\Sigma] = \bigsqcup_{v \in \text{Box}(\Sigma)} \mathcal{X}(\Sigma/\sigma(\bar{v})) .$$

From now on consider $\mathcal{X} = \mathcal{X}_k = [Z_{\bar{\Sigma}_k}/G_{\bar{\Sigma}_k}]$. One can show that the cardinality of $\text{Box}(\bar{\Sigma}_k)$ is $k(2\tilde{k} - 1)$. Moreover, its elements are classified as follows. The element 0 belongs to $\text{Box}(\sigma)$ for every 2-dimensional cone $\sigma \in \{\sigma_1, \dots, \sigma_k, \sigma_{\infty, k+1}, \sigma_{\infty, k+2}\}$. Its corresponding minimal cone is $\{0\} \in \Sigma(0)$. Thus $\mathcal{X}(\Sigma/\{0\}) \simeq \mathcal{X}_k$. Moreover, $\text{Box}(\bar{\Sigma}_k)$ contains $k - 1$ elements of the form $v_\infty, 2v_\infty, \dots, (k-1)v_\infty$ which belong to $\rho_\infty \setminus 0$, thus their corresponding minimal cone is ρ_∞ . Thus for $g_i \in G_{\bar{\Sigma}_k}$ corresponding to iv_∞ for $i = 1, \dots, k-1$, we have and isomorphism $\kappa_i: [Z_{\bar{\Sigma}_k}^{g_i}/G_{\bar{\Sigma}_k}] \xrightarrow{\sim} \mathcal{X}(\bar{\Sigma}_k/\rho_\infty) = \mathcal{D}_\infty$.

Let $i = 1, \dots, k-1$. In the following, we denote by \mathcal{D}_∞^i the substack $[Z_{\bar{\Sigma}_k}^{g_i}/G_{\bar{\Sigma}_k}] \subset \mathcal{I}(\mathcal{X}_k)$. After fixing a primitive k -root of unity ω , it is easy to see that the element g_i is $(1, \dots, 1, \omega^i) \in G_{\bar{\Sigma}_k}$. Then for a scheme S , the objects of $\mathcal{D}_\infty^i(S)$ are pairs of the form (x, g_i) , where x is an object of $\mathcal{D}_\infty(S)$. The case $i = 0$ is excluded because the pairs $(x, 1)$ with $x \in \mathcal{D}_\infty$ are in $\mathcal{X}_k \subset \mathcal{I}(\mathcal{X}_k)$. Moreover, the group of automorphisms of (x, g_i) is μ_k and the inclusion of μ_k into $G_{\bar{\Sigma}_k}$ is given by the map

$$\mu_k \xrightarrow{\gamma_k^i} G_{\bar{\Sigma}_k} = (\mathbb{C}^*)^k, \quad \omega \mapsto g_i .$$

The isomorphism κ_i implies the following commutative triangle

$$\begin{array}{ccc} & \mu_k & \\ \varphi_k^i \swarrow & & \searrow \gamma_k^i \\ \mathbb{C}^* \times \mu_k & \xrightarrow{\iota} & (\mathbb{C}^*)^k \end{array}$$

where the maps φ_k^i and ι are given by:

- for k even:

$$\begin{aligned} \varphi_k^i: \mu_k &\rightarrow \mathbb{C}^* \times \mu_k, & \omega &\mapsto (\omega^i, \omega^{i\tilde{k}}), \\ \iota: \mathbb{C}^* \times \mu_k &\rightarrow (\mathbb{C}^*)^k, & (t, \omega) &\mapsto (1, \dots, 1, t^{\tilde{k}}\omega^{-1}, t). \end{aligned}$$

- for k odd:

$$\begin{aligned} \varphi_k^i: \mu_k &\rightarrow \mathbb{C}^* \times \mu_k, & \omega &\mapsto (\omega^i, 1), \\ \iota: \mathbb{C}^* \times \mu_k &\rightarrow (\mathbb{C}^*)^k, & (t, \omega) &\mapsto (1, \dots, 1, t^k\omega^{\frac{k-1}{2}}, t). \end{aligned}$$

Finally, $\text{Box}(\Sigma)$ contains $k\tilde{k}$ elements which belong to $\sigma_{\infty, k+1}$. Among them, there are exactly k elements which belong to ρ_∞ . Their minimal cone is ρ_∞ . The minimal cone of the other $k\tilde{k} - k$ elements is $\sigma_{\infty, k+1}$. The corresponding group elements are $h_j = (1, \dots, 1, \eta^{j2\tilde{k}}, \eta^j) \in G_{\bar{\Sigma}_k}$ for $j = 0, \dots, k\tilde{k} - 1$, where η is primitive $k\tilde{k}$ -root of unity. For $\tilde{k} \mid j$ we have $h_j = g_{j/\tilde{k}}$ and therefore $[Z_{\bar{\Sigma}_k}^{h_j}/G_{\bar{\Sigma}_k}] \simeq \mathcal{D}_\infty^{j/\tilde{k}}$. So from now on we consider only elements h_j with $j = 1, \dots, k\tilde{k} - 1$, $\tilde{k} \nmid j$. Then for any h_j we have an isomorphism

$\kappa_j^\infty : [Z_{\bar{\Sigma}_k}^{h_j}/G_{\bar{\Sigma}_k}] \xrightarrow{\sim} \mathcal{X}(\bar{\Sigma}_k/\sigma_{\infty,k+1}) = p_\infty$. Let $j = 1, \dots, k\tilde{k} - 1$, $\tilde{k} \nmid j$. Denote by p_∞^j the substack $[Z_{\bar{\Sigma}_k}^{h_j}/G_{\bar{\Sigma}_k}] \subset \mathcal{I}(\mathcal{X}_k)$. Then for a scheme S , the objects of $p_\infty^j(S)$ are pairs of the form (y, h_j) , where $y \in p_\infty(S)$. Moreover, the group of automorphisms of (y, h_j) is $\mu_{k\tilde{k}}$ and the inclusion of $\mu_{k\tilde{k}}$ into $G_{\bar{\Sigma}_k}$ is given by the map

$$\mu_{k\tilde{k}} \xrightarrow{\gamma_j^{k,\infty}} G_{\bar{\Sigma}_k} = (\mathbb{C}^*)^k, \quad \eta \mapsto h_j.$$

The isomorphism κ_j^∞ implies the following commutative triangle

$$\begin{array}{ccc} & \mu_{k\tilde{k}} & \\ \varphi_j^{k,\infty} \swarrow & & \searrow \gamma_j^{k,\infty} \\ \mu_{k\tilde{k}} & \xrightarrow{j^\infty} \mathbb{C}^* \times \mu_k & \xrightarrow{i} (\mathbb{C}^*)^k \end{array}$$

where the maps $\varphi_j^{k,\infty}$ and j^∞ are given by:

- for k even:

$$\begin{aligned} \varphi_j^{k,\infty} : \mu_{k\tilde{k}} &\rightarrow \mu_{k\tilde{k}}, & \eta &\mapsto \eta^j, \\ j^\infty : \mu_{k\tilde{k}} &\rightarrow \mathbb{C}^* \times \mu_k, & \eta &\mapsto (\eta, \eta^{-\tilde{k}}). \end{aligned}$$

- for k odd:

$$\begin{aligned} \varphi_j^{k,\infty} : \mu_{k\tilde{k}} &\rightarrow \mu_{k\tilde{k}}, & \eta &\mapsto \eta^j, \\ j^\infty : \mu_{k\tilde{k}} &\rightarrow \mathbb{C}^* \times \mu_k, & \eta &\mapsto (\eta, \eta^{-2k}). \end{aligned}$$

In a similar way, we obtain substacks $p_0^j \subset \mathcal{I}(\mathcal{X}_k)$ associated to $f_j = (1, \dots, 1, \eta^j) \in G_{\bar{\Sigma}_k}$, which are isomorphic to p_0 , where η is a primitive $k\tilde{k}$ -root of unity and $j = 1, \dots, k\tilde{k} - 1$, $\tilde{k} \nmid j$. Therefore we get a commutative triangle as before

$$\begin{array}{ccc} & \mu_{k\tilde{k}} & \\ \varphi_j^{k,0} \swarrow & & \searrow \gamma_j^{k,0} \\ \mu_{k\tilde{k}} & \xrightarrow{j^0} \mathbb{C}^* \times \mu_k & \xrightarrow{i} (\mathbb{C}^*)^k \end{array}$$

where the maps $\varphi_j^{k,0}$ and j^0 are given by:

- for k even:

$$\begin{aligned} \varphi_j^{k,0} : \mu_{k\tilde{k}} &\rightarrow \mu_{k\tilde{k}}, & \eta &\mapsto \eta^j, \\ j^0 : \mu_{k\tilde{k}} &\rightarrow \mathbb{C}^* \times \mu_k, & \eta &\mapsto (\eta, \eta^{\tilde{k}}). \end{aligned}$$

- for k odd:

$$\begin{aligned} \varphi_j^{k,0} : \mu_{k\tilde{k}} &\rightarrow \mu_{k\tilde{k}}, & \eta &\mapsto \eta^j, \\ j^0 : \mu_{k\tilde{k}} &\rightarrow \mathbb{C}^* \times \mu_k, & \eta &\mapsto (\eta, \eta^{2k}). \end{aligned}$$

Thus we can write the inertia stack in the following way:

$$(142) \quad \mathcal{I}(\mathcal{X}_k) = \mathcal{X}_k \sqcup \left(\bigsqcup_{i=1}^{k-1} \mathcal{D}_\infty^i \right) \sqcup \left(\bigsqcup_{\substack{j=1 \\ \tilde{k} \nmid j}}^{\tilde{k}\tilde{k}-1} p_0^j \right) \sqcup \left(\bigsqcup_{\substack{j=1 \\ \tilde{k} \nmid j}}^{\tilde{k}\tilde{k}-1} p_\infty^j \right).$$

3.2. Topological invariants of \mathcal{X}_k and \mathcal{D}_∞

In this section we compute the integrals of Chern classes of the tangent bundles to \mathcal{X}_k and \mathcal{D}_∞ , which will be useful in the computation of the dimension of the moduli space.

Let us denote by $\mathcal{T}_{\mathcal{D}_\infty}$ the tangent sheaf to \mathcal{D}_∞ . Its canonical bundle is $\omega_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(-p_0 - p_\infty)$. This can be seen as a generalization of the analogous result for varieties [33, Theorem 8.2.3] (cf. [64]). Then by Corollary 4.32 we obtain

$$\omega_{\mathcal{D}_\infty} \simeq \mathcal{L}_1^{\otimes -2\tilde{k}}.$$

By Lemma 3.43, we get

$$(143) \quad \int_{\mathcal{D}_\infty} c_1(\mathcal{T}_{\mathcal{D}_\infty}) = \int_{\mathcal{D}_\infty} c_1(\mathcal{O}_{\mathcal{D}_\infty}(p_0 + p_\infty)) = \frac{2}{k\tilde{k}}.$$

This computation can be done also by using [104, Theorem 3.4] and the two results agree.

By applying [104, Theorem 3.4] to \mathcal{X}_k , we have

$$\int_{\mathcal{I}(\mathcal{X}_k)} c(\mathcal{T}_{\mathcal{I}(\mathcal{X}_k)}) = \int_{\bar{X}_k} c^{SM}(\bar{X}_k) = e(\bar{X}_k) = |\Sigma(2)| = k + 2,$$

where $c^{SM}(\bar{X}_k)$ denotes the Chern-Schwartz-Macpherson class. The second identity comes from [95], and the third from [33, Theorem 12.3.9]. On the other hand, by the decomposition (142) of the inertia stack $\mathcal{I}(\mathcal{X}_k)$, we have

$$\int_{\mathcal{I}(\mathcal{X}_k)} c(\mathcal{T}_{\mathcal{I}(\mathcal{X}_k)}) = \int_{\mathcal{X}_k} c_2(\mathcal{T}_{\mathcal{I}(\mathcal{X}_k)}) + (k-1) \int_{\mathcal{D}_\infty} c_1(\mathcal{T}_{\mathcal{D}_\infty}) + k(\tilde{k}-1) \int_{p_0} 1 + k(\tilde{k}-1) \int_{p_\infty} 1.$$

Recall that the order of the stabilizers of p_0 and p_∞ is $k\tilde{k}$, so that $\int_{p_0} 1 = \frac{1}{k\tilde{k}} \int_{pt} 1 = \frac{1}{k\tilde{k}}$ where pt is intended to be the one-point scheme, so it is the coarse moduli space of p_0 . For p_∞ one obtains the same result, so that

$$(144) \quad \int_{\mathcal{X}_k} c_2(\mathcal{T}_{\mathcal{X}_k}) = k + \frac{2}{k\tilde{k}}.$$

One can compute the previous quantity by using a conjectural analog of the Euler sequence (cf. [33, Theorem 8.1.6]).

3.3. The computation of the Euler characteristic

In this section we collect the results described so far and compute all the ingredients needed to prove Theorem C.1. By using the Töen-Riemann-Roch theorem we have

$$\chi(\mathcal{X}_k, \mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \int_{\mathcal{I}(\mathcal{X}_k)} \frac{\text{ch}(\rho(\pi^*(\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))))}{\text{ch}(\rho(\lambda_{-1}(N^\vee)))} \cdot \text{Td}(\mathcal{T}_{\mathcal{I}(\mathcal{X}_k)}).$$

Using the decomposition (142), the integral over the inertia stack becomes a sum of the following four terms.

$$\begin{aligned}
A &:= \int_{\mathcal{X}_k} \text{ch}(\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \text{Td}(\mathcal{X}_k) , \\
B &:= \sum_{i=1}^{k-1} \int_{\mathcal{D}_\infty^i} \frac{\text{ch}\left(\rho\left((\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty^i}\right)\right)}{\text{ch}\left(\rho\left(\lambda_{-1}(N_{\mathcal{D}_\infty^i/\mathcal{X}_k}^\vee)\right)\right)} \text{Td}(\mathcal{D}_\infty) , \\
C &:= \sum_{\substack{i=1,\dots,k-1 \\ \tilde{k} \nmid i}} \int_{p_0^i} \frac{\text{ch}\left(\rho\left((\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_0^i}\right)\right)}{\text{ch}\left(\rho\left(\lambda_{-1}(N_{p_0^i/\mathcal{X}_k}^\vee)\right)\right)} \text{Td}(p_0) , \\
D &:= \sum_{\substack{i=1,\dots,k-1 \\ \tilde{k} \nmid i}} \int_{p_\infty^i} \frac{\text{ch}\left(\rho\left((\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_\infty^i}\right)\right)}{\text{ch}\left(\rho\left(\lambda_{-1}(N_{p_\infty^i/\mathcal{X}_k}^\vee)\right)\right)} \text{Td}(p_0) .
\end{aligned}$$

We compute each term separately.

3.3.1. Computation of A. Since $\text{ch}(\mathcal{E}') = \text{ch}(\mathcal{E})$, we denote

$$\begin{aligned}
r &:= r(\mathcal{E}') = r(\mathcal{E}) = r(\mathcal{E}^\vee) , \\
\text{ch}_1 &:= \text{ch}_1(\mathcal{E}') = \text{ch}_1(\mathcal{E}) = -\text{ch}_1(\mathcal{E}^\vee) , \\
\text{ch}_2 &:= \text{ch}_2(\mathcal{E}') = \text{ch}_2(\mathcal{E}) = \text{ch}_2(\mathcal{E}^\vee) .
\end{aligned}$$

Moreover,

$$\text{ch}(\mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = 1 - [\mathcal{D}_\infty] + \frac{1}{2}[\mathcal{D}_\infty]^2 .$$

Then we obtain

$$A = \int_{\mathcal{X}_k} (2r\text{ch}_2 - \text{ch}_1^2) + r^2 \int_{\mathcal{X}_k} \left(\text{Td}_2(\mathcal{X}_k) + \frac{1}{2}[\mathcal{D}_\infty]^2 - [\mathcal{D}_\infty]\text{Td}_1(\mathcal{X}_k) \right) .$$

Using equation (144), Proposition 4.13 and adjunction formula [90, Theorem 3.8], we get

$$\begin{aligned}
A &= \int_{\mathcal{X}_k} (2r\text{ch}_2 - \text{ch}_1^2) + r^2 \frac{k^2 \tilde{k}^2 + 4\tilde{k}^2 - 6\tilde{k} + 1}{12 k \tilde{k}^2} \\
&= -2r\Delta + r^2 \frac{k^2 \tilde{k}^2 + 4\tilde{k}^2 - 6\tilde{k} + 1}{12 k \tilde{k}^2} .
\end{aligned}$$

3.3.2. Computation of B. Note first that

$$(\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty} \simeq \mathcal{F}_\infty^{s,\vec{w}} \otimes (\mathcal{F}_\infty^{s,\vec{w}})^\vee \otimes \mathcal{L}_1^{\otimes -1} .$$

Define the translation of the vector \vec{w} in the following way:

$$\begin{aligned}
\vec{w}(0) &= \vec{w} , \\
\vec{w}(i) &= (w_i, \dots, w_{k-1}, w_0, \dots, w_{i-1}) \quad \text{for } i = 1, \dots, k-1 .
\end{aligned}$$

We have

$$\mathcal{F}_\infty^{s, \vec{w}} \otimes (\mathcal{F}_\infty^{s, \vec{w}})^\vee \otimes \mathcal{L}_1^{\otimes -1} \simeq \left(\bigoplus_{j=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(0, j)^{\oplus \vec{w}(0) \cdot \vec{w}(j)} \right) \otimes \mathcal{L}_1^{\otimes -1} \simeq \bigoplus_{j=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(-1, j)^{\oplus \vec{w}(0) \cdot \vec{w}(j)} .$$

Then

$$\rho \left((\mathcal{E}' \otimes \mathcal{E}^\vee \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty^i} \right) = \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \rho_i(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) .$$

LEMMA C.4.

$$\rho_i(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) = \begin{cases} \omega^{i(\tilde{k}j-1)} [\mathcal{O}_{\mathcal{D}_\infty}(-1, j)] & \text{for } k \text{ even} , \\ \omega^{-i} [\mathcal{O}_{\mathcal{D}_\infty}(-1, j)] & \text{for } k \text{ odd} . \end{cases}$$

PROOF. Fix k even. Recall that $\mathcal{O}_{\mathcal{D}_\infty}(-1, j) \simeq \mathcal{L}_1^{\otimes -1} \otimes \mathcal{L}_2^{\otimes j}$ corresponds to the character $\chi^{(-1, j)}: (t, \omega) \in \mathbb{C}^* \times \mu_k \mapsto t^{-1}\omega^j \in \mathbb{C}^*$. The element $\rho_i(\mathcal{O}_{\mathcal{D}_\infty}(-1, j))$ is computed with respect to the map $\varphi_k^i: \omega \in \mu_k \mapsto (\omega^i, \omega^{i\tilde{k}}) \in \mathbb{C}^* \times \mu_k$, where ω is a primitive k -root of unity. So the composition of the latter map with $\chi^{(-1, j)}$ gives

$$\omega \in \mu_k \mapsto \omega^{i(\tilde{k}j-1)} \in \mathbb{C}^* .$$

For k odd one has a similar result. In that case the map φ_k^i is given by $\omega \in \mu_k \mapsto (\omega^i, 1) \in \mathbb{C}^* \times \mu_k$, which by composition with $\chi^{(-1, j)}$ yields

$$\omega \in \mu_k \mapsto \omega^{-i} \in \mathbb{C}^* .$$

□

By applying the previous lemma, we obtain

$$\begin{aligned} \text{ch} \left(\rho \left((\mathcal{E}' \otimes \mathcal{E}^\vee \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty^i} \right) \right) &= \\ &= \begin{cases} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \omega^{i(\tilde{k}j-1)} \text{ch}(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) & \text{for } k \text{ even}, \\ \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \omega^{-i} \text{ch}(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) & \text{for } k \text{ odd}. \end{cases} \end{aligned}$$

The normal bundle $N_{\mathcal{D}_\infty/\mathcal{X}_k}$ is isomorphic to $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}$. Thus by Lemma C.4 we get

$$\text{ch} \left(\rho \left(\lambda_{-1} N_{\mathcal{D}_\infty/\mathcal{X}_k}^\vee \right) \right) = \text{ch} \left(\rho_i (1 - \rho_i(\mathcal{L}_1^{\otimes -1})) \right) = 1 - \omega^{-i} \text{ch}(\mathcal{L}_1^{\otimes -1}) = 1 - \omega^{-i} (1 - c_1(\mathcal{L}_1)) .$$

To invert this class, note that if $x^2 = 0$ (we are not interested in classes of degree greater than 1), then $\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2}$. For every $j = 0, \dots, k-1$ we set $s_j = \tilde{k}j - 1$ if k is even and $s_j = -1$ if k is odd. Then we have

$$\begin{aligned} B &= \sum_{i=1}^{k-1} \int_{\mathcal{D}_\infty} \left[\sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \omega^{is_j} (1 + c_1(\mathcal{O}_{\mathcal{D}_\infty}(-1, j))) \right] \cdot \\ &\quad \cdot \left[\frac{1}{1 - \omega^{-i}} - \frac{\omega^{-i}}{(1 - \omega^{-i})^2} c_1(\mathcal{L}_1) \right] \cdot (1 + \text{Td}_1(\mathcal{D}_\infty)) = \\ &= \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{k-1} \frac{\omega^{is_j}}{1 - \omega^{-i}} \left[\frac{1}{k\tilde{k}} - \frac{1}{k\tilde{k}^2} - \frac{1}{k\tilde{k}^2} \cdot \frac{\omega^{-i}}{1 - \omega^{-i}} \right] , \end{aligned}$$

where we used equation (143) and the fact that $\int_{\mathcal{D}_\infty} c_1(\mathcal{O}_{\mathcal{D}_\infty}(a, j)) = \frac{a}{kk^2}$. Now we use the following identity [26]

$$(145) \quad \frac{1}{k} \sum_{i=1}^{k-1} \frac{\omega^{is}}{1 - \omega^{-i}} = \left\lfloor \frac{s}{k} \right\rfloor - \frac{s}{k} + \frac{k-1}{2k},$$

together with the fact that

$$\left\lfloor \frac{s_j}{k} \right\rfloor - \frac{s_j}{k} = \begin{cases} \frac{1}{\tilde{k}} - 1 & \text{for } k \text{ odd, or } k \text{ even and } j \text{ even,} \\ \frac{1}{\tilde{k}} - \frac{1}{2} & \text{for } k \text{ even and } j \text{ odd.} \end{cases}$$

and $\sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) = r^2$. Thus for k odd we get easily that

$$\begin{aligned} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{k-1} \frac{\omega^{is_j}}{1 - \omega^{-i}} \left[\frac{1}{k\tilde{k}} - \frac{1}{k\tilde{k}^2} \right] &= \\ &= \frac{\tilde{k}-1}{\tilde{k}^2} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \left[\left\lfloor \frac{s_j}{k} \right\rfloor - \frac{s_j}{k} + \frac{k-1}{2k} \right] = -\frac{(k-1)^2}{2k^3} r^2. \end{aligned}$$

Now consider the case k even. Define for the vector \vec{w} the natural numbers $r_e = \sum_{i \text{ even}} w_i$ and $r_o = \sum_{i \text{ odd}} w_i$. Then $r = r_e + r_o$, and $\sum_{j \text{ even}} \vec{w}(0) \cdot \vec{w}(j) = r_e^2 + r_o^2$, thus $\sum_{j \text{ odd}} \vec{w}(0) \cdot \vec{w}(j) = 2r_e r_o$. Then for k even we have

$$\frac{\tilde{k}-1}{\tilde{k}^2} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \left[\left\lfloor \frac{s_j}{k} \right\rfloor - \frac{s_j}{k} + \frac{k-1}{2k} \right] = (r_e^2 + r_o^2) \frac{(\tilde{k}-1)(1-k)}{2k\tilde{k}^2} + 2r_e r_o \frac{\tilde{k}-1}{2k\tilde{k}^2}.$$

Using also Lemma F.1 for k odd and F.2 for k even, we get

$$B = \begin{cases} -\frac{5k^2-6k+1}{12k^3} r^2 & \text{for } k \text{ odd;} \\ -\frac{5k^2+12k-4}{12k^3} r^2 - \frac{(r_e-r_o)^2}{4k} & \text{for } k \text{ even.} \end{cases}$$

Adding the expression we obtained for A , we get

$$(146) \quad A + B = \begin{cases} -2r\Delta + \frac{k^2-1}{12k} r^2 - \frac{(r_e-r_o)^2}{4k} & \text{for } k \text{ even;} \\ -2r\Delta + \frac{k^2-1}{12k} r^2 & \text{for } k \text{ odd.} \end{cases}$$

3.3.3. Computation of C and D . Consider p_0 . We have

$$\begin{aligned} (\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_0} &\simeq \left(\mathcal{F}_\infty^{s,\vec{w}} \otimes (\mathcal{F}_\infty^{s,\vec{w}})^\vee \otimes \mathcal{L}_1^{\otimes -1} \right)|_{p_0} \simeq \\ &\simeq \left(\bigoplus_{j=0}^{k-1} (\mathcal{O}_{\mathcal{D}_\infty}(-1, j))^{\oplus \vec{w}(0) \cdot \vec{w}(j)} \right)|_{p_0} \simeq \bigoplus_{j=0}^{k-1} (\mathcal{L}_{p_0}^{\otimes \tilde{k}j-1})^{\oplus \vec{w}(0) \cdot \vec{w}(j)}. \end{aligned}$$

Then for every $i = 1, \dots, k\tilde{k}-1$ such that $\tilde{k} \nmid i$, we get

$$\text{ch}_0 \left(\rho \left((\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_0^i} \right) \right) = \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \eta^{i(\tilde{k}j-1)}.$$

LEMMA C.5. *The conormal bundle to p_0 in \mathcal{X}_k has the form*

$$N_{p_0/\mathcal{X}_k}^\vee \simeq \mathcal{L}_{p_0}^{\otimes -2\tilde{k}} \oplus \mathcal{L}_{p_0}^{\otimes -1}$$

PROOF. Being p_0 a 0-dimensional substack in \mathcal{X}_k , its tangent bundle is trivial, so that $N_{p_0/\mathcal{X}_k}^\vee \simeq \mathcal{T}_{\mathcal{X}_k|p_0}^\vee$. The divisors \mathcal{D}_0 and \mathcal{D}_∞ , which intersect in p_0 , are normal crossing, so that the tangent bundle splits as

$$\mathcal{T}_{\mathcal{X}_k|p_0} \simeq \mathcal{T}_{\mathcal{D}_0|p_0} \oplus \mathcal{T}_{\mathcal{D}_\infty|p_0}.$$

By adjunction formula, we obtain

$$\begin{aligned} \mathcal{T}_{\mathcal{D}_0|p_0} &\simeq (\omega_{\mathcal{D}_0}^\vee)_{|p_0} \simeq \left((\omega_{\mathcal{X}_k} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0))_{|\mathcal{D}_\infty}^\vee \right)_{|p_0} \\ &\simeq \left(\mathcal{O}_{\mathcal{X}_k} \left(- \sum_{i=1, \dots, k, \infty} \mathcal{D}_i \right)_{|\mathcal{D}_\infty} \right)_{|p_0}^\vee \simeq \mathcal{L}_{1|p_0} \simeq \mathcal{L}_{p_0}. \end{aligned}$$

Since $\omega_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(-p_0 - p_\infty)$, we get

$$\mathcal{T}_{\mathcal{D}_\infty|p_0} \simeq (\mathcal{O}_{\mathcal{D}_\infty}(p_0 + p_\infty))_{|p_0} \simeq \mathcal{L}_1^{\otimes 2\tilde{k}}|_{p_0} \simeq \mathcal{L}_{p_0}^{\otimes 2\tilde{k}},$$

and the statement is proved. \square

By using the previous lemma we obtain

$$\text{ch}_0 \left(\rho \left(\lambda_{-1} (N_{p_0/\mathcal{X}_k}^\vee) \right) \right) = (1 - \eta^{-i})(1 - \eta^{-2i\tilde{k}}),$$

and therefore

$$\begin{aligned} C &= \sum_{\substack{i=1, \dots, k\tilde{k}-1 \\ \tilde{k} \nmid i}} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \frac{\eta^{i(\tilde{k}j-1)}}{(1 - \eta^{-i})(1 - \eta^{-2i\tilde{k}})} \int_{p_0} 1 = \\ &= \frac{1}{k\tilde{k}} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{\substack{i=1, \dots, k\tilde{k}-1 \\ \tilde{k} \nmid i}} \frac{\eta^{i(\tilde{k}j-1)}}{(1 - \eta^{-i})(1 - \eta^{-2i\tilde{k}})}. \end{aligned}$$

By doing a similar computation for p_∞ , we obtain

$$(\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))_{|p_\infty} \simeq \bigoplus_{j=0}^{k-1} (\mathcal{L}_{p_\infty}^{\otimes -\tilde{k}j-1})^{\oplus \vec{w}(0) \cdot \vec{w}(j)},$$

then

$$\text{ch}_0 \left(\rho \left((\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))_{|p_\infty} \right) \right) = \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \eta^{-i(\tilde{k}j+1)}.$$

By arguing as in the proof of the previous lemma, one can prove the following result.

LEMMA C.6. *The conormal bundle to p_∞ in \mathcal{X}_k has the form*

$$N_{p_\infty/\mathcal{X}_k}^\vee \simeq \mathcal{L}_{p_0}^{\otimes -2\tilde{k}} \oplus \mathcal{L}_{p_0}^{\otimes -1}.$$

By the previous lemma, we get

$$\text{ch}_0 \left(\rho \left(\lambda_{-1}(N_{p_\infty^i/\mathcal{X}_k}^\vee) \right) \right) = (1 - \eta^{-i})(1 - \eta^{-2i\tilde{k}}) ,$$

and thus

$$D = \frac{1}{k\tilde{k}} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{\substack{i=1, \dots, k\tilde{k}-1 \\ \tilde{k} \nmid i}} \frac{\eta^{-i(\tilde{k}j+1)}}{(1 - \eta^{-i})(1 - \eta^{-2i\tilde{k}})} .$$

Finally we obtain

$$C + D = \frac{1}{k\tilde{k}} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{\substack{i=1, \dots, k\tilde{k}-1 \\ \tilde{k} \nmid i}} \frac{\eta^{i(\tilde{k}j-1)} + \eta^{-i(\tilde{k}j+1)}}{(1 - \eta^{-i})(1 - \eta^{-2i\tilde{k}})} .$$

Now we have to distinguish two cases.

3.3.3.1. *k odd.* By using Lemma F.6 we have

$$\begin{aligned} C + D &= \frac{1}{k^2} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{k-1} \frac{\omega^{ij} + \omega^{-ij}}{1 - \omega^{-2i}} \sum_{l=0}^{k-1} \frac{1}{\eta^i \omega^l - 1} \\ &= \frac{1}{4k} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{k-1} (\omega^{ij} + \omega^{-ij}) \left(\frac{3 - \omega^i}{(1 - \omega^{-i})^2} + \frac{\omega^{2i}}{1 + \omega^i} \right) . \end{aligned}$$

It is convenient to separate the contributions from $j = 0$ and $j \geq 1$ in the above sum; we call the corresponding contributions $(C + D)_0$ and $(C + D)_>$, respectively. By Lemma F.4 and Lemma F.7 we easily find

$$(C + D)_0 = -\frac{k^2 - 1}{12k} \vec{w}(0)^2$$

where $\vec{w}(0)^2 := \sum_{i=0}^{k-1} w_i^2$, while by Lemma F.5 and Lemma F.8 we get

$$(C + D)_> = \sum_{j=1}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \left(\frac{j(k-j)}{2k} - \frac{k^2 - 1}{12k} \right) .$$

Thus

$$(147) \quad C + D = -\frac{k^2 - 1}{12k} r^2 + \sum_{j=1}^{k-1} \frac{j(k-j)}{2k} \vec{w}(0) \cdot \vec{w}(j) .$$

3.3.3.2. *k even.* By using Lemma F.9 we have

$$\begin{aligned}
C + D &= \frac{2}{k^2} \sum_{j=0}^{k-1} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{\tilde{k}-1} \frac{\omega^{ij} + \omega^{-ij}}{1 - \omega^{-2i}} \sum_{l=0}^{k-1} \frac{(-1)^l j}{\eta^i \omega^l - 1} = \\
&\quad \frac{2}{k} \sum_{j \text{ even}} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{\tilde{k}-1} \frac{\omega^{i(j-2)} + \omega^{i(-j-2)}}{(1 - \omega^{-2i})^2} + \\
&\quad + \frac{2}{k} \sum_{j \text{ odd}} \vec{w}(0) \cdot \vec{w}(j) \sum_{i=1}^{\tilde{k}-1} \frac{\omega^{i(j-1)} + \omega^{i(-j-1)}}{(1 - \omega^{-2i})^2} .
\end{aligned}$$

For j even, we set $p = j/2$ and we use Lemma F.10 with p and $\tilde{k} - p$ instead of j , while for j odd we set $q = (j+1)/2$ and use again Lemma F.10 with q and $\tilde{k} - q + 1$. This gives

$$(148) \quad C + D = -\frac{k^2 - 1}{12k} r^2 + \sum_{j=1}^{k-1} \frac{j(k-j)}{2k} \vec{w}(0) \cdot \vec{w}(j) + \frac{(r_e - r_o)^2}{4k} .$$

3.4. Dimension formula

Now we prove Theorem C.1. Assume that there exist points $[(\mathcal{E}, \phi_{\mathcal{E}})]$ and $[(\mathcal{E}', \phi'_{\mathcal{E}})]$ in the moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}})$ such that \mathcal{E} and \mathcal{E}' are locally free sheaves. By Proposition 5.7, we have

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}}) &= \dim_{\mathbb{C}} \text{Ext}^1(\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_{\infty})) = \\
&= -\chi(\mathcal{E}^{\vee} \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_{\infty})) .
\end{aligned}$$

Thus $\dim_{\mathbb{C}} \mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}}) = -(A + B + C + D)$, and by Formule (146), (148), (147) we get

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{s, \vec{w}}) &= 2r\Delta - \sum_{j=1}^{k-1} \frac{j(k-j)}{2k} \vec{w}(0) \cdot \vec{w}(j) = \\
(149) \quad &= 2r\Delta - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{j,j} \vec{w}(0) \cdot \vec{w}(j) .
\end{aligned}$$

REMARK C.7. As we saw in Section 5.1.2, $\mathcal{M}_{1, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{D}_{\infty}})$ is isomorphic to $\text{Hilb}^{\Delta}(X_k)$. In the rank one case one has $\vec{w}(0) \cdot \vec{w}(j) = 0$ for all $j \geq 1$ and Formula (149) agrees with the dimension of $\text{Hilb}^{\Delta}(X_k)$.

Likewise, when $w_i = r$ for some $i \in \{0, 1, \dots, k-1\}$ and $w_j = 0$ for all $j \neq i$, the dimension is given by

$$2r \Delta = 2r n + (r-1) \vec{v} \cdot C \vec{v} .$$

△

EXAMPLE C.8. Set $k = 2$. In this case, $\vec{w} = (w_0, w_1)$ and $r = w_0 + w_1$, and the dimension formula reduces to

$$\chi(\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_2}(-\mathcal{D}_\infty)) = -2r\Delta + \frac{r^2}{8} - \frac{(w_0 - w_1)^2}{8} = -2r\Delta + \frac{w_0 w_1}{2}.$$

This number is actually an integer. To show this fix a locally free sheaf

$$(150) \quad \mathcal{E} := \oplus_{i=1}^r \mathcal{R}_1^{\otimes u_i} \otimes \mathcal{O}_{\mathcal{X}_2}(\mathcal{D}_\infty)^{\otimes s}$$

on \mathcal{X}_2 . Note that

$$\mathcal{E}|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(s, 0)^{\oplus w_0} \oplus \mathcal{O}_{\mathcal{D}_\infty}(s, 1)^{\oplus w_1},$$

where $w_0 := \#\{i \mid u_i \text{ even}\}$ and $w_1 := \#\{i \mid u_i \text{ odd}\}$. Denote by $\phi_{\mathcal{E}}$ the previous isomorphism. Then $(\mathcal{E}, \phi_{\mathcal{E}})$ is a $(\mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ -framed sheaf of rank $r = w_0 + w_1$. Moreover,

$$\begin{aligned} \det(\mathcal{E}) &\simeq \mathcal{R}_1^{\otimes u} \otimes \mathcal{O}_{\mathcal{X}_2}(\mathcal{D}_\infty)^{\otimes sr}, \\ \text{ch}_2(\mathcal{E}) &= \frac{1}{4}(rs^2 - \sum_{i=1}^r u_i^2), \end{aligned}$$

where $u := \sum_{i=1}^r u_i$. Thus

$$(151) \quad \chi(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_2}(-\mathcal{D}_\infty)) = -2r\Delta + \frac{w_0 w_1}{2} = -\frac{r}{2} \sum_{i=1}^r u_i^2 + \frac{1}{2}(u^2 + w_0 w_1).$$

Note that the quantity $u^2 + w_0 w_1$ is always even, hence $\chi(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_2}(-\mathcal{D}_\infty))$ is an integer.

On the other hand, if we fix the vector \vec{w} and integers s, u, m , there exists a locally free sheaf \mathcal{E} of the form (150) if and only if there exists a decomposition $u = \sum_{i=1}^{w_0+w_1} u_i$ of u such that $\sum_{i=1}^{w_0+w_1} u_i^2 = rs^2 - m$.

Let us consider the following particular choice of the locally free sheaf \mathcal{E} of the form (150):

$$\mathcal{E} := \oplus_{i=1}^r \mathcal{R}_1^{\otimes u_i},$$

with all u_i even integer numbers. This choice implies $s = 0$ and $w_1 = 0$. Then formula (151) becomes

$$(152) \quad \chi(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_2}(-\mathcal{D}_\infty)) = -\frac{r}{2} \sum_{i=1}^r u_i^2 + \frac{1}{2}u^2.$$

Set $u_i = 2v_i$ for $i = 1, \dots, r$. The locally free sheaf $\pi_{2*}(\mathcal{E})$ is

$$\pi_{2*}(\mathcal{E}) = \pi_{2*}(\oplus_{i=1}^r \mathcal{R}_1^{\otimes u_i}) = \oplus_{i=1}^r \mathcal{O}_{\mathbb{F}_2}(2v_i D_0 - v_i D_\infty),$$

because $\mathcal{R}_1 \simeq \mathcal{O}_{\mathcal{X}_2}(\mathcal{D}_0 - \mathcal{D}_\infty)$. Moreover, the Euler characteristic $\chi(\pi_{2*}(\mathcal{E})^\vee \otimes \pi_{2*}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{F}_2}(-D_\infty))$ is exactly (152). \triangle

EXAMPLE C.9. For $k = 3$ the dimension formula reads

$$\dim_{\mathbb{C}} \mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_3, \mathcal{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) = 2r\Delta - \frac{w_0 w_1 + w_1 w_2 + w_2 w_0}{6}.$$

\triangle

APPENDIX D

The edge contribution

In this appendix we prove Proposition 5.17 and Corollary 5.20 and give explicit expressions for the terms $L_{\alpha\beta}^l$ and $\ell_{\alpha\beta}^{(l)}$.

Recall that we have to compute explicitly

$$L_{\alpha,\beta}(t_1, t_2) = -\chi_{T_t}(\mathcal{X}_k, \mathcal{R}^{\vec{u}_\beta - \vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$$

In the following we compute $-\chi(\mathcal{X}_k, \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ for an arbitrary vector $\vec{u} \in \mathbb{Z}^{k-1}$.

4.1. Generalities

LEMMA D.1. *Given a vector $\vec{u} \in \mathbb{Z}^{k-1}$, for every $j = 1, \dots, k-1$ there is an exact sequence*

$$(153) \quad 0 \rightarrow \mathcal{R}^{\vec{u} + C\vec{e}_j} \rightarrow \mathcal{R}^{\vec{u}} \rightarrow \mathcal{R}_{|\mathcal{D}_j}^{\vec{u}} \rightarrow 0 ,$$

where C is the Cartan matrix of type A_{k-1} .

PROOF. Fix $j = 1, \dots, k-1$ and consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_j) \rightarrow \mathcal{O}_{\mathcal{X}_k} \rightarrow \mathcal{O}_{\mathcal{D}_j} \rightarrow 0 .$$

We obtain the assertion just by tensoring the previous sequence by $\mathcal{R}^{\vec{u}}$. Indeed, we need only to prove that $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_j) = \mathcal{R}^{\vec{u} + C\vec{e}_j}$. By definition $\mathcal{R}^{\vec{u}} = \mathcal{O}_{\mathcal{X}_k}(\sum_{i=1}^{k-1} u_i \omega_i)$, so we have

$$\begin{aligned} \sum_{i=1}^{k-1} u_i \omega_i - \mathcal{D}_j &= \sum_{i=1}^{k-1} u_i \omega_i - \sum_{l=1}^{k-1} \sum_{i=1}^{k-1} C_{ji}(C^{-1})^{il} \mathcal{D}_l = \\ &= \sum_{i=1}^{k-1} u_i \omega_i + \sum_{i=1}^{k-1} C_{ji} \omega_i = \sum_{i=1}^{k-1} (u_i + (C\vec{e}_j)_i) \omega_i . \end{aligned}$$

□

LEMMA D.2. *Let $\vec{u} \in \mathbb{Z}^{k-1}$ and $j = 1, \dots, k-1$. Then*

$$(154) \quad \mathcal{R}_{|\mathcal{D}_j}^{\vec{u}} \simeq \mathcal{O}_{\mathcal{D}_j}(u_j) .$$

PROOF. Since $\mathcal{D}_j \xrightarrow{\pi_{k|\mathcal{D}_j}} D_j \simeq \mathbb{P}^1$ for $j = 1, \dots, k-1$, the Picard group $\text{Pic}(\mathcal{D}_j)$ of \mathcal{D}_j is a free abelian group generated by the line bundle $\mathcal{O}_{\mathcal{D}_j}(1) := \pi_{k|\mathcal{D}_j}^* \mathcal{O}_{D_j}(1)$. So $\mathcal{R}_{|\mathcal{D}_j}^{\vec{u}} \simeq$

$\mathcal{O}_{\mathcal{D}_j}(1)^{\otimes a_j} =: \mathcal{O}_{\mathcal{D}_j}(a_j)$ for some integer a_j . Thus a_j is the degree of $\mathcal{R}_{|\mathcal{D}_j}^{\vec{u}}$, and to determine a_j it is enough to compute $\sum_{i=1} u_i \omega_i \cdot \mathcal{D}_j$. By using the relation (45), we get

$$\begin{aligned} \sum_{i=1} u_i \omega_i \cdot \mathcal{D}_j &= - \sum_{i=1}^{k-1} u_i \sum_{l=1}^{k-1} (C^{-1})^{il} \mathcal{D}_l \cdot \mathcal{D}_j \\ &= - \sum_{i=1}^{k-1} u_i \left((C^{-1})^{ij-1} \mathcal{D}_{j-1} + (C^{-1})^{ij} \mathcal{D}_j + (C^{-1})^{ij+1} \mathcal{D}_{j+1} \right) \cdot \mathcal{D}_j = u_j . \end{aligned}$$

□

4.2. The induction

Set $c \in \{0, \dots, k-1\}$ to be the equivalence class modulo k of $k(C^{-1}\vec{u})_{k-1}$. Define $\vec{u}' := \vec{u} - \vec{e}_c$ if $c > 0$, $\vec{u}' := \vec{u}$ otherwise, where \vec{e}_c is the c -th coordinate vector of \mathbb{Z}^{k-1} . Define also $\vec{v} := C^{-1}\vec{u}'$. Then by construction, $\vec{v} \in \mathbb{Z}^{k-1}$.

Now we can start the induction procedure. To obtain it we shall use Lemma D.1 to simplify the computations. For $c = 0$ we set the convention $\mathcal{R}_c := \mathcal{O}_{\mathcal{X}_k}$. Consider first the case when $v_i \geq 0$ for every i . By using v_1 times the exact sequence (153) for $i = 1$, we obtain

$$\begin{aligned} \chi(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) &= \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}'} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \\ &= \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C\vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C\vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_1}) \\ &\quad \vdots \\ &= \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C v_1 \vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \sum_{j=1}^{v_1} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C j \vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_1}) . \end{aligned}$$

Now we do other v_2 steps with the sequence (153) for $i = 2$ and we obtain

$$\begin{aligned} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C v_1 \vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) &= \\ &= \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C v_1 \vec{e}_1 - C \vec{e}_2} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C v_1 \vec{e}_1 - C \vec{e}_2} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_2}) \\ &\quad \vdots \\ &= \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(v_1 \vec{e}_1 + v_2 \vec{e}_2)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) + \\ &\quad - \sum_{j=1}^{v_2} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(v_1 \vec{e}_1 + j \vec{e}_2)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_2}) . \end{aligned}$$

Iterating this procedure using v_i times the sequence (153) for $i = 3, \dots, k-1$, we get

$$\begin{aligned}
\chi(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) &= \\
&= \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C\vec{v}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) + \\
&\quad - \sum_{i=1}^{k-1} \sum_{j=1}^{v_i} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(\sum_{p=1}^{i-1} v_p \vec{e}_p + j \vec{e}_i)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_i}) = \\
&= \chi(\mathcal{R}_c \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \sum_{i=1}^{k-1} \sum_{j=1}^{v_i} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(\sum_{p=1}^{i-1} v_p \vec{e}_p + j \vec{e}_i)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_i}) .
\end{aligned}$$

If one of the v_i 's is negative, one can follow the procedure described before by using $-v_i$ times the short exact sequence (153). In this case, one exchanges the roles played into the procedure by the left and middle terms of the sequence.

Let us define the L factors as

$$L_{\vec{u}}^l := \begin{cases} -\sum_{i=1}^{v_l} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(\sum_{p=1}^{l-1} v_p \vec{e}_p + i \vec{e}_l)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_l}) & \text{for } v_l \geq 0 \\ \sum_{i=0}^{-v_l-1} \chi(\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(\sum_{p=1}^{l-1} v_p \vec{e}_p - i \vec{e}_l)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_l}) & \text{for } v_l < 0 . \end{cases}$$

Then we obtain

$$\chi(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \chi(\mathcal{R}_c \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) + \sum_{l=1}^{k-1} L_{\vec{u}}^l .$$

By Theorem E.1 in Appendix E, we have

$$(155) \quad \chi(\mathcal{R}_c \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = 0$$

for every $c = 0, 1, \dots, k-1$. Thus remains just to compute the L factors. By Lemma D.2, we have

$$\begin{aligned}
\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(\sum_{p=1}^{l-1} v_p \vec{e}_p + j \vec{e}_l)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_l} &\simeq \mathcal{O}_{\mathcal{D}_l}(\delta_{l,c} + u'_l + v_{l-1} - 2j) \\
\mathcal{R}_c \otimes \mathcal{R}^{\vec{u}' - C(\sum_{p=1}^{l-1} v_p \vec{e}_p - j \vec{e}_l)} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)|_{\mathcal{D}_l} &\simeq \mathcal{O}_{\mathcal{D}_l}(\delta_{l,c} + u'_l + v_{l-1} + 2j) .
\end{aligned}$$

Then, recalling that $u'_l = (C\vec{v})_l = -v_{l-1} + 2v_l - v_{l+1}$, we can rewrite the L -factors as

$$L_{\vec{u}}^l = \begin{cases} -\sum_{j=0}^{v_l-1} \chi(\mathcal{O}_{\mathcal{D}_l}(\delta_{l,c} - v_{l+1} + 2j)) & \text{for } v_l \geq 0 , \\ \sum_{j=1}^{-v_l} \chi(\mathcal{O}_{\mathcal{D}_l}(\delta_{l,c} - v_{l+1} - 2j)) & \text{for } v_l < 0 . \end{cases}$$

EXAMPLE D.3. Let $k = 2$. Then $c \in \{0, 1\}$ and

$$L_{\vec{u}}^1 = \begin{cases} -\sum_{j=0}^{v_1-1} \chi(\mathcal{O}_{\mathcal{D}_1}(\delta_{1,c} + 2j)) & \text{for } v_1 \geq 0 , \\ \sum_{j=1}^{-v_1} \chi(\mathcal{O}_{\mathcal{D}_1}(\delta_{1,c} - 2j)) & \text{for } v_1 < 0 . \end{cases}$$

△

EXAMPLE D.4. Let $k = 3$. Then for $c \in \{0, 1, 2\}$ we get

$$L_{\vec{u}}^1 := \begin{cases} -\sum_{j=0}^{v_1-1} \chi(\mathcal{O}_{\mathcal{D}_1}(\delta_{1,c} - v_2 + 2j)) & \text{for } v_1 \geq 0 , \\ = \sum_{j=1}^{-v_1} \chi(\mathcal{O}_{\mathcal{D}_1}(\delta_{1,c} - v_2 - 2j)) & \text{for } v_1 < 0 . \end{cases}$$

and

$$L_u^2 := \begin{cases} -\sum_{j=0}^{v_2-1} \chi(\mathcal{O}_{\mathcal{D}_2}(\delta_{2,c} + 2j)) & \text{for } v_2 \geq 0, \\ \sum_{j=1}^{-v_2} \chi(\mathcal{O}_{\mathcal{D}_2}(\delta_{2,c} - 2j)) & \text{for } v_2 < 0. \end{cases}$$

△

4.3. Characters of the restrictions and final results

Here we choose the equivariant structure on $\mathcal{O}_{\mathcal{D}_l}(a)$ given by the isomorphism

$$(156) \quad \mathcal{O}_{\mathcal{D}_l}(a) \simeq \mathcal{O}_{\mathcal{X}_k} \left(-\left\lfloor \frac{a}{2} \right\rfloor \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right)_{|\mathcal{D}_l}.$$

The reason for this choice is that it makes the computations easier, and it is possible to give a closed formula for the edge contribution.

THEOREM D.5. *Fix $l \in \{1, \dots, k-1\}$. We have for $a \geq 0$,*

$$\begin{aligned} \chi(\mathcal{O}_{\mathcal{D}_l}(a)) &= \left(\chi_1^l \right)^{\left\lfloor \frac{a}{2} \right\rfloor} \sum_{j=0}^a \left(\chi_2^l \right)^j, \\ \chi(\mathcal{O}_{\mathcal{D}_l}(-a)) &= - \left(\chi_1^l \right)^{-\left\lfloor \frac{a}{2} \right\rfloor} \sum_{j=1}^{a-1} \left(\chi_2^l \right)^{-j} \end{aligned}$$

PROOF. Let $a \geq 0$ and consider the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathcal{X}_k} \left(\left(-\left\lfloor \frac{a}{2} \right\rfloor - 1 \right) \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right) &\rightarrow \mathcal{O}_{\mathcal{X}_k} \left(-\left\lfloor \frac{a}{2} \right\rfloor \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right) \rightarrow \\ &\rightarrow \mathcal{O}_{\mathcal{X}_k} \left(-\left\lfloor \frac{a}{2} \right\rfloor \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right)_{|\mathcal{D}_l} \rightarrow 0. \end{aligned}$$

Then for the Euler characteristic we have

$$\begin{aligned} \chi \left(\mathcal{O}_{\mathcal{X}_k} \left(-\left\lfloor \frac{a}{2} \right\rfloor \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right)_{|\mathcal{D}_l} \right) &= \\ = \chi \left(\mathcal{O}_{\mathcal{X}_k} \left(-\left\lfloor \frac{a}{2} \right\rfloor \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right) \right) - \chi \left(\mathcal{O}_{\mathcal{X}_k} \left(\left(-\left\lfloor \frac{a}{2} \right\rfloor - 1 \right) \mathcal{D}_l + 2 \left\{ \frac{a}{2} \right\} \mathcal{D}_{l+1} \right) \right) &= \\ = \chi \left(\mathcal{O}_{X_k} \left(-\left\lfloor \frac{a}{2} \right\rfloor D_l + 2 \left\{ \frac{a}{2} \right\} D_{l+1} \right) \right) - \chi \left(\mathcal{O}_{X_k} \left(\left(-\left\lfloor \frac{a}{2} \right\rfloor - 1 \right) D_l + 2 \left\{ \frac{a}{2} \right\} D_{l+1} \right) \right). \end{aligned}$$

To conclude the proof it is sufficient to compute, for $m \geq 0$,

$$\begin{aligned} &\chi(\mathcal{O}_{X_k}(-mD_l)) - \chi(\mathcal{O}_{X_k}(-(m+1)D_l)), \\ &\chi(\mathcal{O}_{X_k}(D_{l+1} - mD_l)) - \chi(\mathcal{O}_{X_k}(D_{l+1} - (m+1)D_l)). \end{aligned}$$

For the first equality, by [33, Proposition 9.1.6], is easy to verify that the zero and second cohomology groups vanishes. Moreover, computing the first cohomology group is equivalent to count the integer points on the line of direction $(l-1, l)$, between the points $-(l-2)m, -(l-$

$1)m)$ and $(lm, (l+1)m)$. We get easily

$$\begin{aligned}\chi(\mathcal{O}_{X_k}(-mD_l)) - \chi(\mathcal{O}_{X_k}(-(m+1)D_l)) &= \sum_{i=0}^{2m} T_1^{-(l-2)m+i(l-1)} T_2^{-(l-1)m+i(l-1)} \\ &= (\chi_1^l)^m \sum_{i=0}^{2m} (\chi_2^l)^i .\end{aligned}$$

In the same way, for the second equality to prove we have

$$\begin{aligned}\chi(\mathcal{O}_{X_k}(D_{l+1} - mD_l)) - \chi(\mathcal{O}_{X_k}(D_{l+1} - (m+1)D_l)) &= \sum_{i=0}^{2m+1} T_1^{-(l-2)m+i(l-1)} T_2^{-(l-1)m+i(l-1)} \\ &= (\chi_1^l)^m \sum_{i=0}^{2m+1} (\chi_2^l)^i .\end{aligned}$$

For $a < 0$ one can argue in the same way. \square

REMARK D.6. Given the first equality in Theorem D.5, one can get the second also by *equivariant* Serre duality. In particular we have, for $a > 0$,

$$\chi(\mathcal{O}_{\mathcal{D}_l}(-a)) = -(\chi_1^l)^{-1}(\chi_2^l)^{-1}(\chi(\mathcal{O}_{\mathcal{D}_l}(a-2)^\vee)) .$$

\triangle

Now we use Theorem D.5 to compute the expression of $L_{\vec{u}}^l$ in (4.2). Fix $l \in \{1, \dots, k-1\}$ and denote $d = d(l, c) := \delta_{l,c} - v_{l+1}$. For $v_l \geq 0$ we have

$$(157) \quad L_{\vec{u}}^l = - \sum_{i=0}^{v_l-1} \chi(\mathcal{O}_{\mathcal{D}_l}(d+2i)) = \begin{cases} - \sum_{i=0}^{v_l-1} \sum_{j=0}^{d+2i} (\chi_1^l)^{\lfloor d/2 \rfloor + i} (\chi_2^l)^j & \text{for } d \geq 0 , \\ \sum_{i=1}^{-\lfloor d/2 \rfloor - 1} \sum_{j=1}^{-2\{d/2\}+2i} (\chi_1^l)^{2\{d/2\}-i} (\chi_2^l)^{-j} + \\ \quad - \sum_{i=0}^{\lfloor d/2 \rfloor + v_l - 1} \sum_{j=0}^{2\{d/2\}+2i} (\chi_1^l)^i (\chi_2^l)^j & \text{for } 2 - 2v_l \leq d < 0 , \\ \sum_{i=0}^{v_l-1} \sum_{j=1}^{-d-2i-1} (\chi_1^l)^{-\lfloor -d/2 \rfloor + i} (\chi_2^l)^{-j} & \text{for } d < 2 - 2v_l . \end{cases}$$

For $v_l < 0$ we have similar expressions:

$$(158) \quad L_{\vec{u}}^l = \sum_{i=1}^{-v_l} \chi(\mathcal{O}_{\mathcal{D}_l}(d-2i)) = \begin{cases} \sum_{i=1}^{-v_l} \sum_{j=1}^{-d+2i-1} (\chi_1^l)^{-\lfloor -d/2 \rfloor - i} (\chi_2^l)^{-j} & \text{for } d < 2 , \\ \sum_{i=1}^{v_l - \lfloor d/2 \rfloor} \sum_{j=1}^{-2\{d/2\}+2i} (\chi_1^l)^{2\{d/2\}-i} (\chi_2^l)^{-j} + \\ \quad - \sum_{i=0}^{\lfloor d/2 \rfloor - 1} \sum_{j=0}^{2\{d/2\}+2i} (\chi_1^l)^i (\chi_2^l)^j & \text{for } 2 \leq d < -2v_l , \\ - \sum_{i=1}^{-v_l} \sum_{j=0}^{d-2i} (\chi_1^l)^{\lfloor d/2 \rfloor - i} (\chi_2^l)^j & \text{for } d \geq -2v_l . \end{cases}$$

EXAMPLE D.7. For $k = 2$ we have just one factor L , and just two possible cases:

$$L_{\vec{u}}^1 = \begin{cases} -\sum_{i=0}^{v_1-1} \sum_{j=0}^{2i+\delta_{1,c}} (\chi_1^1)^i (\chi_2^1)^j & \text{for } v_1 \geq 0, \\ \sum_{i=1}^{-v_1} \sum_{j=1}^{2i-1-\delta_{1,c}} (\chi_1^1)^{-i} (\chi_2^1)^{-j} & \text{for } v_1 < 0. \end{cases}$$

△

EXAMPLE D.8. For $k = 3$ we start seeing all the possible cases for L_1 : for $v_1 \geq 0$ we have

$$L_{\vec{u}}^1 = \begin{cases} -\sum_{i=0}^{v_1-1} \sum_{j=0}^{\delta_{1,c}-v_2+2i} (\chi_1^1)^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor + i} (\chi_2^1)^j & \text{for } \delta_{1,c} - v_2 \geq 0, \\ \sum_{i=1}^{-\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - 1} \sum_{j=1}^{-2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^{2\{\frac{\delta_{1,c}-v_2}{2}\} - i} (\chi_2^1)^{-j} + \\ -\sum_{i=0}^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor + v_1 - 1} \sum_{j=0}^{2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^i (\chi_2^1)^j & \text{for } 2 - 2v_1 \leq \delta_{1,c} - v_2 < 0, \\ \sum_{i=0}^{v_1-1} \sum_{j=1}^{-d-2i-1} (\chi_1^1)^{-\lfloor -d/2 \rfloor + i} (\chi_2^1)^{-j} & \text{for } \delta_{1,c} - v_2 < 2 - 2v_1. \end{cases}$$

For $v_1 < 0$ we have similar expressions:

$$L_{\vec{u}}^1 = \begin{cases} \sum_{i=1}^{-v_1} \sum_{j=1}^{v_2-\delta_{1,c}+2i-1} (\chi_1^1)^{-\lfloor \frac{v_2-\delta_{1,c}}{2} \rfloor - i} (\chi_2^1)^{-j} & \text{for } \delta_{1,c} - v_2 < 2, \\ \sum_{i=1}^{v_1 - \lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor} \sum_{j=1}^{-2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^{2\{\frac{\delta_{1,c}-v_2}{2}\} - i} (\chi_2^1)^{-j} + \\ -\sum_{i=0}^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - 1} \sum_{j=0}^{2\{\frac{\delta_{1,c}-v_2}{2}\} + 2i} (\chi_1^1)^i (\chi_2^1)^j & \text{for } 2 \leq \delta_{1,c} - v_2 < -2v_1, \\ -\sum_{i=1}^{-v_1} \sum_{j=0}^{\delta_{1,c}-v_2-2i} (\chi_1^1)^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - i} (\chi_2^1)^j & \text{for } \delta_{1,c} - v_2 \geq -2v_1. \end{cases}$$

For L_2 it simplifies to

$$L_{\vec{u}}^2 = \begin{cases} -\sum_{i=0}^{v_2-1} \sum_{j=0}^{2i+\delta_{2,c}} (\chi_1^2)^i (\chi_2^2)^j & \text{for } v_2 \geq 0, \\ \sum_{i=1}^{-v_2} \sum_{j=1}^{2i-1-\delta_{2,c}} (\chi_1^2)^{-i} (\chi_2^2)^{-j} & \text{for } v_2 < 0. \end{cases}$$

△

By taking the Euler class, one also get, for $v_l \geq 0$

$$(159) \quad \ell^{(l)}(\varepsilon_1, \varepsilon_2) = \begin{cases} \prod_{i=0}^{v_l-1} \prod_{j=0}^{d+2i} \left((\lfloor d/2 \rfloor + i) \varepsilon_1^{(l)} + j \varepsilon_2^{(l)} \right)^{-1} & \text{for } d \geq 0, \\ \prod_{i=1}^{-\lfloor d/2 \rfloor - 1} \prod_{j=1}^{-2\{d/2\} + 2i} \left((2\{d/2\} - i) \varepsilon_1^{(l)} - j \varepsilon_2^{(l)} \right) \cdot \\ \cdot \prod_{i=0}^{\lfloor d/2 \rfloor + v_l - 1} \prod_{j=0}^{2\{d/2\} + 2i} \left(i \varepsilon_1^{(l)} + j \varepsilon_2^{(l)} \right)^{-1} & \text{for } 2 - 2v_l \leq d < 0, \\ \prod_{i=0}^{v_l-1} \prod_{j=1}^{-d-2i-1} \left((-\lfloor -d/2 \rfloor + i) \varepsilon_1^{(l)} - j \varepsilon_2^{(l)} \right) & \text{for } d < 2 - 2v_l. \end{cases}$$

For $v_l < 0$ we get

$$(160) \quad \ell^{(l)}(\varepsilon_1, \varepsilon_2) = \begin{cases} \prod_{i=1}^{-v_l} \prod_{j=1}^{-d+2i-1} \left((-\lfloor -d/2 \rfloor - i) \varepsilon_1^{(l)} - j \varepsilon_2^{(l)} \right) & \text{for } d < 2, \\ \prod_{i=1}^{v_l - \lfloor d/2 \rfloor} \prod_{j=1}^{-2\{d/2\}+2i} \left((2\{d/2\} - i) \varepsilon_1^{(l)} - j \varepsilon_2^{(l)} \right) \cdot \\ \quad \cdot \prod_{i=0}^{\lfloor d/2 \rfloor - 1} \prod_{j=0}^{2\{d/2\}+2i} \left(i \varepsilon_1^{(l)} + j \varepsilon_2^{(l)} \right)^{-1} & \text{for } 2 \leq d < -2v_l, \\ \prod_{i=1}^{-v_l} \prod_{j=0}^{d-2i} \left((\lfloor d/2 \rfloor - i) \varepsilon_1^{(l)} + j \varepsilon_2^{(l)} \right)^{-1} & \text{for } d \geq -2v_l. \end{cases}$$

EXAMPLE D.9. For $k = 2$ we have

$$\ell^{(1)}(\varepsilon_1, \varepsilon_2) = \begin{cases} \prod_{i=0}^{v_1-1} \prod_{j=0}^{2i+\delta_{1,c}} \left(i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1} & \text{for } v_1 \geq 0, \\ \prod_{i=1}^{-v_1} \prod_{j=1}^{2i-1-\delta_{1,c}} \left(-i \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} \right) & \text{for } v_1 < 0. \end{cases}$$

△

EXAMPLE D.10. For $k = 3$ we have $\ell^{(1)}$ and $\ell^{(2)}$. For the first, with $v_l \geq 0$

$$\ell^{(1)}(\varepsilon_1, \varepsilon_2) = \begin{cases} \prod_{i=0}^{v_1-1} \prod_{j=0}^{\delta_{1,c}-v_2+2i} \left(\left(\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor + i \right) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1} & \text{for } \delta_{1,c} - v_2 \geq 0, \\ \prod_{i=1}^{-\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - 1} \prod_{j=1}^{-2\{\frac{\delta_{1,c}-v_2}{2}\}+2i} \left(\left(2\{\frac{\delta_{1,c}-v_2}{2}\} - i \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} \right) \cdot \\ \quad \cdot \prod_{i=0}^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor + v_1 - 1} \prod_{j=0}^{2\{\frac{\delta_{1,c}-v_2}{2}\}+2i} \left(i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1} & \text{for } 2 - 2v_1 \leq \delta_{1,c} - v_2 < 0, \\ \prod_{i=0}^{v_1-1} \prod_{j=1}^{-d-2i-1} \left((-\lfloor -d/2 \rfloor + i) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} \right) & \text{for } \delta_{1,c} - v_2 < 2 - 2v_1. \end{cases}$$

For $v_1 < 0$ we have

$$\ell^{(1)}(\varepsilon_1, \varepsilon_2) = \begin{cases} \prod_{i=1}^{-v_1} \prod_{j=1}^{v_2-\delta_{1,c}+2i-1} \left(\left(-\left\lfloor \frac{v_2-\delta_{1,c}}{2} \right\rfloor - i \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} \right) & \text{for } \delta_{1,c} - v_2 < 2, \\ \prod_{i=1}^{v_1 - \lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor} \prod_{j=1}^{-2\{\frac{\delta_{1,c}-v_2}{2}\}+2i} \left(\left(2\{\frac{\delta_{1,c}-v_2}{2}\} - i \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} \right) \cdot \\ \quad \cdot \prod_{i=0}^{\lfloor \frac{\delta_{1,c}-v_2}{2} \rfloor - 1} \prod_{j=0}^{2\{\frac{\delta_{1,c}-v_2}{2}\}+2i} \left(i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1} & \text{for } 2 \leq \delta_{1,c} - v_2 < -2v_1, \\ \prod_{i=1}^{-v_1} \prod_{j=0}^{\delta_{1,c}-v_2-2i} \left(\left(\left\lfloor \frac{\delta_{1,c}-v_2}{2} \right\rfloor - i \right) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1} & \text{for } \delta_{1,c} - v_2 \geq -2v_1. \end{cases}$$

$\ell^{(2)}$ simplifies to

$$\ell^{(2)}(\varepsilon_1, \varepsilon_2) = \begin{cases} \prod_{i=0}^{v_2-1} \prod_{j=0}^{2i+\delta_{2,c}} \left(i \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1} & \text{for } v_2 \geq 0, \\ \prod_{i=1}^{-v_2} \prod_{j=1}^{2i-1-\delta_{2,c}} \left(-i \varepsilon_1^{(1)} - j \varepsilon_2^{(1)} \right) & \text{for } v_2 < 0. \end{cases}$$

△

APPENDIX E

Vanishing theorems for tautological line bundles

In this appendix we use the Töen-Riemann-Roch theorem and the identities on complex roots of unity discussed in Appendix F to prove a vanishing theorem for the tautological line bundles \mathcal{R}_j . First we state the result.

THEOREM E.1. *For $j = 0, \dots, k-1$ we have*

$$\chi(\mathcal{X}_k, \mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = 0 ,$$

where we denoted $\mathcal{R}_0 = \mathcal{O}_{\mathcal{X}_k}$.

By the Töen-Riemann-Roch theorem we have

$$\chi(\mathcal{X}_k, \mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \int_{\mathcal{I}(\mathcal{X}_k)} \frac{\text{ch}(\rho(\pi^*(\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))))}{\text{ch}(\rho(\lambda_{-1}(N^\vee)))} \cdot \text{Td}(\mathcal{T}_{\mathcal{I}(\mathcal{X}_k)}) .$$

Using the decomposition (142), the integral over the inertia stack becomes a sum of the following four terms.

$$\begin{aligned} A &:= \int_{\mathcal{X}_k} \text{ch}(\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \text{Td}(\mathcal{X}_k) , \\ B &:= \sum_{i=1}^{k-1} \int_{\mathcal{D}_\infty^i} \frac{\text{ch}\left(\rho\left((\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty^i}\right)\right)}{\text{ch}\left(\rho\left(\lambda_{-1}(N_{\mathcal{D}_\infty^i/\mathcal{X}_k}^\vee)\right)\right)} \text{Td}(\mathcal{D}_\infty) , \\ C &:= \sum_{\substack{i=1, \dots, k-1 \\ \tilde{k} \nmid i}} \int_{p_0^i} \frac{\text{ch}\left(\rho\left((\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_0^i}\right)\right)}{\text{ch}\left(\rho\left(\lambda_{-1}(N_{p_0^i/\mathcal{X}_k}^\vee)\right)\right)} \text{Td}(p_0) , \\ D &:= \sum_{\substack{i=1, \dots, k-1 \\ \tilde{k} \nmid i}} \int_{p_\infty^i} \frac{\text{ch}\left(\rho\left((\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_\infty^i}\right)\right)}{\text{ch}\left(\rho\left(\lambda_{-1}(N_{p_\infty^i/\mathcal{X}_k}^\vee)\right)\right)} \text{Td}(p_0) . \end{aligned}$$

We compute each term separately.

Computation of A.

$$\begin{aligned} A &= \int_{\mathcal{X}_k} c_1(\mathcal{R}_j) \left(\frac{1}{2} c_1(\mathcal{R}_j) - [\mathcal{D}_\infty] + \text{Td}_1(\mathcal{X}_k) \right) + \\ &\quad + \int_{\mathcal{X}_k} \left(\text{Td}_2(\mathcal{X}_k) + \frac{1}{2} [\mathcal{D}_\infty]^2 - [\mathcal{D}_\infty] \text{Td}_1(\mathcal{X}_k) \right) \\ &= -\frac{1}{2} (C^{-1})^{j,j} + \frac{1}{2} \int_{\mathcal{X}_k} c_1(\mathcal{R}_j) c_1(\mathcal{T}_{\mathcal{X}_k}) + \frac{k^2 \tilde{k}^2 + 4\tilde{k}^2 - 6\tilde{k} + 1}{12 k \tilde{k}^2} , \end{aligned}$$

where we used Formula (48) and, for the second integral, the computation of A in Appendix C. Now we have the following.

LEMMA E.2. *For every $j = 1, \dots, k-1$*

$$\int_{\mathcal{X}_k} c_1(\mathcal{R}_j) c_1(\mathcal{T}_{\mathcal{X}_k}) = 0 .$$

PROOF. We already know that $c_1(\mathcal{T}_{\mathcal{X}_k}) = -c_1(\omega_{\mathcal{X}_k}) = \sum_{i=0, \dots, k, \infty} [\mathcal{D}_i]$. Using the description (45) and the intersection product between the $[\mathcal{D}_i]$'s, it is easy to see that

$$\int_{\mathcal{X}_k} c_1(\mathcal{R}_j) [\mathcal{D}_i] = \delta_{i,j} \quad \text{for } i = 1, \dots, k-1 .$$

Moreover in the same way we have

$$\int_{\mathcal{X}_k} c_1(\mathcal{R}_j) [\mathcal{D}_0] = \frac{i-k}{k} \quad \int_{\mathcal{X}_k} c_1(\mathcal{R}_j) [\mathcal{D}_k] = -\frac{i}{k} \quad \int_{\mathcal{X}_k} c_1(\mathcal{R}_j) [\mathcal{D}_\infty] = 0 .$$

Thus,

$$\begin{aligned} \int_{\mathcal{X}_k} c_1(\mathcal{R}_j) c_1(\mathcal{T}_{\mathcal{X}_k}) &= \sum_{i=0, \dots, k, \infty} \int_{\mathcal{X}_k} c_1(\mathcal{R}_j) [\mathcal{D}_i] \\ &= \frac{j-k}{k} + \sum_{i=1, \dots, k-1} \delta_{i,j} - \frac{j}{k} = 0 . \end{aligned}$$

□

Summing up, we obtained

$$(161) \quad A = -\frac{j(k-j)}{2k} + \frac{k^2 \tilde{k}^2 + 4\tilde{k}^2 - 6\tilde{k} + 1}{12 k \tilde{k}^2} .$$

Computation of B . Note that $(\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(-1, j)$, so by Lemma C.4, we obtain

$$\text{ch} \left(\rho \left((\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{\mathcal{D}_\infty} \right) \right) = \begin{cases} \omega^{i(\tilde{k}j-1)} \text{ch}(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) & \text{for } k \text{ even} , \\ \omega^{-i} \text{ch}(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) & \text{for } k \text{ odd} . \end{cases}$$

The normal bundle $N_{\mathcal{D}_\infty/\mathcal{X}_k}$ is isomorphic to $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}$. Thus by Lemma C.4 we get

$$\text{ch} \left(\rho \left(\lambda_{-1} N_{\mathcal{D}_\infty/\mathcal{X}_k}^\vee \right) \right) = \text{ch} \left(\rho_i (1 - \rho_i(\mathcal{L}_1^{\otimes -1})) \right) = 1 - \omega^{-i} \text{ch}(\mathcal{L}_1^{\otimes -1}) = 1 - \omega^{-i} (1 - c_1(\mathcal{L}_1)) .$$

Set $s_j = \tilde{k}j - 1$ if k is even and $s_j = -1$ if k is odd. Therefore, we obtain

$$\begin{aligned} B &= \sum_{i=1}^{k-1} \int_{\mathcal{D}_\infty} [\omega^{is_j} (1 + c_1(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)))] \cdot \left[\frac{1}{1 - \omega^{-i}} - \frac{\omega^{-i}}{(1 - \omega^{-i})^2} c_1(\mathcal{L}_1) \right] \cdot (1 + \text{Td}_1(\mathcal{D}_\infty)) \\ &= \frac{\tilde{k} - 1}{k \tilde{k}^2} \sum_{i=1}^{k-1} \frac{\omega^{is_j}}{1 - \omega^{-i}} - \frac{1}{k \tilde{k}^2} \frac{\omega^{is_j-1}}{(1 - \omega^{-i})^2} , \end{aligned}$$

where the last equality is given by equation (143) and by $\int_{\mathcal{D}_\infty} c_1(\mathcal{O}_{\mathcal{D}_\infty}(a, j)) = \frac{a}{k \tilde{k}^2}$.

Now we follow the computation of B in Appendix C. Set first k odd; using the identity [26] and Lemma F.1, we obtain

$$\begin{aligned} B &= \frac{k-1}{k^3} \sum_{i=1}^{k-1} \frac{\omega^{-i}}{1-\omega^{-i}} - \frac{1}{k^3} \frac{\omega^{-2i}}{(1-\omega^{-i})^2} \\ &= -\frac{5k^2-6k+1}{12k^3}. \end{aligned}$$

In the same way, for k even, using again the identity [26] and Lemma F.2, we obtain

$$B = \begin{cases} -\frac{2k^2-3k+1}{3k^3} & \text{for } j \text{ even ,} \\ -\frac{k^2-6k+2}{6k^3} & \text{for } j \text{ odd .} \end{cases}$$

Summing with formula (161) we have

$$(162) \quad A + B = \begin{cases} -\frac{j(k-j)}{2k} + \frac{k^2-1}{12k} & \text{for } k \text{ odd ,} \\ -\frac{j(k-j)}{2k} + \frac{k^2-4}{12k} & \text{for } k \text{ even, } j \text{ even ,} \\ -\frac{j(k-j)}{2k} + \frac{k^2+2}{12k} & \text{for } k \text{ even, } j \text{ odd .} \end{cases}$$

Computation of C and D . Since we have

$$\begin{aligned} (\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_0} &\simeq \mathcal{L}_{p_0}^{\otimes \tilde{k}j-1}, \\ (\mathcal{R}_j \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|_{p_\infty} &\simeq \mathcal{L}_{p_\infty}^{\otimes -\tilde{k}j-1}, \end{aligned}$$

repeating the computations for $C + D$ in Appendix C, in particular for Lemmas C.5 and C.6, we get

$$C + D = \frac{1}{k\tilde{k}} \sum_{\substack{i=1, \dots, k\tilde{k}-1 \\ \tilde{k} \nmid i}} \frac{\eta^{i(\tilde{k}j-1)} + \eta^{-i(\tilde{k}j+1)}}{(1-\eta^{-i})(1-\eta^{-2i\tilde{k}})}.$$

Set k odd. By the same computations as in Section 3.3.3.1, we have

$$C + D = \frac{1}{4k} \sum_{i=1}^{k-1} (\omega^{ij} + \omega^{-ij}) \left(\frac{3 - \omega^i}{(1 - \omega^{-i})^2} + \frac{\omega^{2i}}{1 + \omega^i} \right).$$

Using, for the three sums, Lemmas F.5 and F.8, one obtains

$$C + D = \frac{j(k-j)}{2k} - \frac{k^2-1}{12k},$$

then by (162), $A + B + C + D = 0$, as stated.

For k even, following Section 3.3.3.2, we obtain

$$C + D = \begin{cases} \frac{2}{k} \sum_{i=1}^{\tilde{k}-1} \frac{\omega^{i(j-2)} + \omega^{i(-j-2)}}{(1-\omega^{-2i})^2} & \text{for } j \text{ even ,} \\ \frac{2}{k} \sum_{i=1}^{\tilde{k}-1} \frac{\omega^{i(j-1)} + \omega^{i(-j-1)}}{(1-\omega^{-2i})^2} & \text{for } j \text{ odd .} \end{cases}$$

Now, for j even, we set $p = j/2$ and we use Lemma F.10 with p and $\tilde{k} - p$ instead of j , while for j odd we set $q = (j+1)/2$ and use again Lemma F.10 with q and $\tilde{k} - q + 1$. This gives

$$C + D = \begin{cases} \frac{j(k-j)}{2k} - \frac{k^2-4}{12k} & \text{for } j \text{ even ,} \\ \frac{j(k-j)}{2k} - \frac{k^2+2}{12k} & \text{for } j \text{ odd ,} \end{cases}$$

thus again by (162), $A + B + C + D = 0$.

APPENDIX F

Identities on complex roots of unity

In this appendix we collect a number of identities developed by R. Szabo that have been used in Appendices C and E. ω will denote a complex k -root of unity, and η a complex $k\tilde{k}$ -root of unity.

6.1. Identities for the B contributions

LEMMA F.1.

$$\sum_{i=1}^{k-1} \frac{\omega^{-2i}}{(1 - \omega^{-i})^2} = -\frac{(k-5)(k-1)}{12} .$$

PROOF. By using the same arguments as in the proof of Formula (145) in [26], one can prove the following identity:

$$(163) \quad \sum_{i=1}^{k-1} \frac{\omega^{is}}{(1 - \omega^{-i})^2} = \sum_{l=0}^{k-1} (s-l) \left(\left\lfloor \frac{s-l}{k} \right\rfloor - \frac{s-l}{k} + \frac{k-1}{2k} \right) ,$$

In our case, from the previous identity, we get

$$\begin{aligned} \sum_{i=1}^{k-1} \frac{\omega^{-2i}}{(1 - \omega^{-i})^2} &= -\sum_{l=0}^{k-1} (l+2) \left(-\left\{ \frac{-l-2}{k} \right\} + \frac{k-1}{2k} \right) \\ &= \sum_{m=1}^{k-1} m \frac{k-m}{k} + k-1 - \frac{k-1}{2k} \sum_{m=2}^{k+1} m . \end{aligned}$$

By doing some algebraic manipulations, one get the assertion. □

LEMMA F.2. *Take k even and $j \in \{0, \dots, k-1\}$. Then*

$$\sum_{i=1}^{k-1} \frac{\omega^{i(\tilde{k}j-2)}}{(1 - \omega^{-i})^2} = \begin{cases} -\frac{(k-5)(k-1)}{12} & \text{for } j \text{ even ;} \\ \frac{k^2-10}{24} & \text{for } j \text{ odd .} \end{cases}$$

PROOF. By using again the identity (163) we obtain

$$\sum_{i=1}^{k-1} \frac{\omega^{i(\tilde{k}j-2)}}{(1 - \omega^{-i})^2} = \sum_{l=0}^{k-1} (\tilde{k}j-2-l) \left(-\left\{ \frac{\tilde{k}j-l-2}{k} \right\} + \frac{k-1}{2k} \right) .$$

With easy computations one shows

$$\begin{aligned} \frac{k-1}{2k} \sum_{l=0}^{k-1} (\tilde{k}j - 2 - l) &= j \left(\frac{k^2 - k}{4} \right) - \frac{k-1}{2k} \left(\sum_{m=1}^{k+1} m - 1 \right) \\ &= j \left(\frac{k^2 - k}{4} \right) - \frac{(k-1)(k+3)}{4}. \end{aligned}$$

The other term is a little more complicated and we have to distinguish two cases. First set j even, then

$$\begin{aligned} - \sum_{l=0}^{k-1} (\tilde{k}j - 2 - l) \left\{ \frac{j}{2} + \frac{-l-2}{k} \right\} &= -\tilde{k}j \sum_{m=2}^{k+1} \left\{ -\frac{m}{k} \right\} + \sum_{m=2}^{k+1} m \left\{ -\frac{m}{k} \right\} \\ &= -j \left(\frac{k^2 - k}{4} \right) + \frac{(k+7)(k-1)}{6}. \end{aligned}$$

By adding the two terms, we get the assertion for j even. For j odd, in the same way

$$- \sum_{l=0}^{k-1} (\tilde{k}j - 2 - l) \left\{ \frac{j}{2} + \frac{-l-2}{k} \right\} = -j \left(\frac{k^2 - k}{4} \right) + \frac{7k^2 + 12k - 28}{24}.$$

Again adding the two terms, we get the assertion for j odd. \square

6.2. Identities for C and D contributions

We divide these identities according to the parity of k .

6.2.1. k odd.

LEMMA F.3. *For any fixed $1 \leq i \leq k-1$ and $x \in \mathbb{C} \setminus \mu_k$, we have*

$$\prod_{\substack{j=1 \\ j \neq i}}^{k-1} (x - \omega^j) = - \sum_{n=0}^{k-2} x^n \sum_{l=1}^{n+1} \omega^{-li}$$

and

$$\sum_{i=1}^{k-1} \frac{1}{x - \omega^i} = \frac{\sum_{n=0}^{k-2} (n+1) x^n}{\sum_{n=0}^{k-1} x^n}.$$

PROOF. By definition, k -th roots of unity are zeroes of the monic polynomial $x^k - 1$, so that

$$x^k - 1 = \prod_{i=0}^{k-1} (x - \omega^i).$$

On the other hand, the elementary geometric series

$$\sum_{n=0}^{k-1} x^n = \frac{x^k - 1}{x - 1}$$

implies that $\sum_{i=0}^{k-1} \omega^{si} = k$ if $s \equiv 0 \pmod k$ and $\sum_{i=0}^{k-1} \omega^{si} = 0$ otherwise, and moreover

$$\prod_{\substack{j=1 \\ j \neq i}}^{k-1} (x - \omega^j) = \frac{1}{x - \omega^i} \sum_{n=0}^{k-1} x^n =: \sum_{n=0}^{k-2} c_n x^n .$$

The polynomial coefficients $n! c_n$ can be obtained by differentiating the second expression n times with respect to x at $x = 0$, and it is straightforward to prove by induction that

$$c_n = - \sum_{l=1}^{n+1} \omega^{-li} .$$

Note in particular that $c_{k-2} = - \sum_{l=1}^{k-1} \omega^{-li} = 1$ as expected.

For the second identity, we write

$$\sum_{i=1}^{k-1} \frac{1}{x - \omega^i} = \frac{\sum_{i=1}^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k-1} (x - \omega^j)}{\prod_{i=1}^{k-1} (x - \omega^i)} .$$

From above we have

$$\prod_{i=1}^{k-1} (x - \omega^i) = \frac{x^k - 1}{x - 1} = \sum_{n=0}^{k-1} x^n$$

and

$$\sum_{i=1}^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k-1} (x - \omega^j) = - \sum_{n=0}^{k-2} x^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-lj} = \sum_{n=0}^{k-2} (n+1) x^n ,$$

and the result follows. \square

LEMMA F.4.
$$\sum_{i=1}^{k-1} \frac{\omega^{2i}}{1 + \omega^i} = - \frac{k+1}{2} .$$

PROOF. Since k is odd, setting $x = -1$ in Lemma F.3 gives

$$\prod_{i=1}^{k-1} (1 + \omega^i) = \sum_{n=0}^{k-1} (-1)^n = 1$$

and

$$\begin{aligned} \sum_{i=1}^{k-1} \omega^{2i} \prod_{\substack{j=1 \\ j \neq i}}^{k-1} (1 + \omega^j) &= \sum_{n=0}^{k-2} (-1)^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-(l-2)i} \\ &= -1 - \sum_{n=1}^{\frac{k-1}{2}} 2n + \sum_{n=1}^{\frac{k-1}{2}} (2n-1) = -\frac{k+1}{2} . \end{aligned}$$

\square

LEMMA F.5. For any $1 \leq j \leq k-1$, one has

$$\sum_{i=1}^{k-1} \frac{\omega^{i(j+2)} + \omega^{-i(j-2)}}{1 + \omega^i} = -1 .$$

PROOF. Putting $x = -1$ in Lemma F.3 again we find

$$\begin{aligned} \sum_{i=1}^{k-1} \frac{\omega^{i(j+2)}}{1 + \omega^i} &= \sum_{n=1}^{k-2} (-1)^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-i(l-j-2)} \\ &= \sum_{n=0}^j (-1)^{n+1} (n+1) + \sum_{n=j+1}^{k-1} (-1)^n (k-1) + \sum_{n=j+1}^{k-1} (-1)^{n+1} n . \end{aligned}$$

For j odd this gives

$$\frac{j+1}{2} + (k-1) - \frac{k-1}{2} - \frac{j+1}{2} = \frac{k-1}{2}$$

while for j even we get

$$-\frac{j}{2} - 1 - \frac{k-1}{2} + \frac{j}{2} = -\frac{k+1}{2} .$$

Now the sum

$$\sum_{i=1}^{k-1} \frac{\omega^{-i(j-2)}}{1 + \omega^i} = \sum_{i=1}^{k-1} \frac{\omega^{i(k-j+2)}}{1 + \omega^i}$$

is computed in an identical way by just replacing j with $k-j$. Since j and $k-j$ have opposite parity, for any $j \in \{1, \dots, k-1\}$ we get

$$\sum_{i=1}^{k-1} \frac{\omega^{i(j+2)} + \omega^{-i(j-2)}}{1 + \omega^i} = \frac{k-1}{2} - \frac{k+1}{2} = -1 .$$

□

LEMMA F.6. Let η be a k -th root of ω , $\eta^k = \omega$, and $1 \leq i \leq k-1$. Then

$$\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{\eta^i \omega^j - 1} = \frac{1}{\omega^i - 1} .$$

PROOF. Using Lemma F.3 with $x = \eta^{-i}$ we compute

$$\begin{aligned} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{\eta^i \omega^j - 1} &= -\frac{\eta^{-i}}{k} \left(\frac{1}{\eta^{-i} - 1} + \frac{\sum_{n=0}^{k-2} (n+1) \eta^{-in}}{\sum_{n=0}^{k-1} \eta^{-in}} \right) \\ &= -\frac{\eta^{-i}}{k} \frac{2\eta^{-i(k-1)} + (k-2)\eta^{-i(k-1)}}{\eta^{-ik} - 1} = \frac{1}{\omega^i - 1} . \end{aligned}$$

□

LEMMA F.7.

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{1}{(1-\omega^{-i})^2} &= -\frac{(k+5)(k+1)}{12} \\ \sum_{i=1}^{k-1} \frac{\omega^i}{(1-\omega^{-i})^2} &= -\frac{k^2-12k+23}{12}\end{aligned}$$

PROOF. Setting $s = 0$ in the identity (163) gives

$$\sum_{i=1}^{k-1} \frac{1}{(1-\omega^{-i})^2} = -\sum_{l=0}^{k-1} l \left(\left\lfloor -\frac{l}{k} \right\rfloor + \frac{l}{k} + \frac{k-1}{2k} \right) = -\sum_{l=0}^{k-1} l \left(\frac{l}{k} - \frac{k+1}{2k} \right),$$

and the result now follows by elementary algebraic manipulations. For the second identity, setting $s = 1$ in (163) gives

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{\omega^i}{(1-\omega^{-i})^2} &= -\frac{1}{k} + \frac{k-1}{2k} - \sum_{l=2}^{k-1} (l-1) \left(\left\lfloor -\frac{l-1}{k} \right\rfloor + \frac{l-1}{k} + \frac{k-1}{2k} \right) \\ &= -\frac{1}{k} + \frac{k-1}{2k} - \sum_{l=1}^{k-2} l \left(\frac{l}{k} - \frac{k+1}{2k} \right),\end{aligned}$$

and the result easily follows. \square

LEMMA F.8. For $1 \leq j \leq k-1$, we have

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{\omega^{ij} + \omega^{-ij}}{(1-\omega^{-i})^2} &= j(k-j) - \frac{k^2+5}{6} \\ \sum_{i=1}^{k-1} \frac{\omega^{i(j+1)} + \omega^{-i(j-1)}}{(1-\omega^{-i})^2} &= j(k-j) - \frac{k^2+23}{6}\end{aligned}$$

PROOF. Setting $s = j$ in (163) we get

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{\omega^{ij}}{(1-\omega^{-i})^2} &= \left(\sum_{l=j}^{k-1} + \sum_{l=0}^{j-1} \right) (j-l) \left(\left\lfloor \frac{j-l}{k} \right\rfloor - \frac{j-l}{k} + \frac{k-1}{2k} \right) \\ &= -\sum_{l=0}^{k-1-j} l \left(\frac{l}{k} - 1 + \frac{k-1}{2k} \right) + \sum_{l=1}^j l \left(-\frac{l}{k} + \frac{k-1}{2k} \right) \\ &= \frac{j(k-j)}{2} - j - \frac{(k-5)(k-1)}{12}.\end{aligned}$$

From this formula, the sum

$$\sum_{i=1}^{k-1} \frac{\omega^{-ij}}{(1-\omega^{-i})^2} = \sum_{i=1}^{k-1} \frac{\omega^{i(k-j)}}{(1-\omega^{-i})^2}$$

can be computed by substituting j with $k - j$, and adding the two sums then gives the result. For the second sum, we set $s = j + 1$ in (163) to get

$$\sum_{i=1}^{k-1} \frac{\omega^{i(j+1)}}{(1 - \omega^{-i})^2} = \left(\sum_{l=0}^j + \sum_{l=j+1}^{k-1} \right) (j - l + 1) \left(\left\lfloor \frac{j - l + 1}{k} \right\rfloor - \frac{j - l + 1}{k} + \frac{k - 1}{2k} \right).$$

For $j = k - 1$ this sum is computed by the first sum of Lemma F.7, while for $j \in \{1, \dots, k - 2\}$ we get

$$\begin{aligned} \sum_{i=1}^{k-1} \frac{\omega^{i(j+1)}}{(1 - \omega^{-i})^2} &= \sum_{l=1}^{j+1} l \left(-\frac{l}{k} + \frac{k-1}{2k} \right) + \sum_{l=0}^{k-j-2} l \left(\frac{l}{k} - \frac{k+1}{2k} \right) \\ &= \frac{j(k-j)}{2} - 2j - \frac{k^2 - 12k + 23}{12}. \end{aligned}$$

Once again the sum

$$\sum_{i=1}^{k-1} \frac{\omega^{-i(j-1)}}{(1 - \omega^{-i})^2} = \sum_{i=1}^{k-1} \frac{\omega^{i(k-j+1)}}{(1 - \omega^{-i})^2}$$

is obtained by replacing j with $k - j$, and adding the two sums finally gives the claimed result. \square

6.2.2. k even.

LEMMA F.9. *Let η be a $k\tilde{k}$ -th root of ω , $\eta^{\tilde{k}} = \omega$, and $1 \leq i \leq \tilde{k} - 1$. Then*

$$\begin{aligned} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{\eta^i \omega^j - 1} &= \frac{1}{\omega^{2i} - 1}, \\ \frac{1}{k} \sum_{j=0}^{k-1} \frac{(-1)^j}{\eta^i \omega^j - 1} &= \frac{\omega^i}{\omega^{2i} - 1}. \end{aligned}$$

PROOF. The first identity follows exactly as in the proof of Lemma F.6, except that now $\eta^{\tilde{k}} = \omega^2$. For the second identity, we proceed as in the proof of Lemma F.6. Using $\omega^{\tilde{k}} = -1$ we first compute

$$\begin{aligned} \sum_{j=1}^{k-1} (-1)^j \sum_{n=0}^{k-2} \eta^{-in} \sum_{l=1}^{n+1} \omega^{-lj} &= - \sum_{n=0}^{\tilde{k}-2} \eta^{-in} (n+1) + \sum_{n=\tilde{k}-1}^{k-2} \eta^{-in} (k-1) - \sum_{n=\tilde{k}-1}^{k-2} \eta^{-in} n \\ &= -\eta^i \sum_{n=0}^{\tilde{k}-1} (n(1 + \omega^{-i}) - \tilde{k} \omega^{-i}) \eta^{-in}. \end{aligned}$$

Using also

$$\sum_{n=0}^{k-1} \eta^{-in} = \left(\sum_{n=0}^{\tilde{k}-1} + \sum_{n=\tilde{k}}^{2\tilde{k}-1} \right) \eta^{-in} = (1 + \omega^{-i}) \sum_{n=0}^{\tilde{k}-1} \eta^{-in},$$

we arrive at

$$\begin{aligned} \frac{1}{k} \sum_{j=0}^{k-1} \frac{(-1)^j}{\eta^i \omega^j - 1} &= -\frac{\eta^{-i}}{k} \left(\frac{1}{\eta^{-i} - 1} + \frac{\eta^i}{1 + \omega^{-i}} \frac{\sum_{n=0}^{\tilde{k}-1} (n(1 + \omega^{-i}) - \tilde{k} \omega^{-i}) \eta^{-in}}{\sum_{n=0}^{\tilde{k}-1} \eta^{-in}} \right) \\ &= -\frac{\eta^{-i}}{k} \frac{-\tilde{k} \omega^{-i} \omega^{-i} - 1}{(\omega^{-i} - 1)(1 + \omega^{-i})} \eta^i + (1 + \omega^{-i}) \tilde{k} \eta^i \omega^{-i} = \frac{\omega^i}{\omega^{2i} - 1}. \end{aligned}$$

□

LEMMA F.10. *For any $0 \leq j \leq \tilde{k} - 1$, we have*

$$\sum_{i=1}^{\tilde{k}-1} \frac{\omega^{2i(j-1)}}{(1 - \omega^{-2i})^2} = \frac{j(k-2j)}{4} - \frac{k^2 - 4}{48}.$$

PROOF. Setting $s = j - 1$ in (163) with \tilde{k} instead of k and ω^2 instead of ω gives

$$\begin{aligned} \sum_{i=1}^{\tilde{k}-1} \frac{\omega^{2i(j-1)}}{(1 - \omega^{-2i})^2} &= \left(\sum_{l=0}^{j-2} + \sum_{l=j-1}^{\tilde{k}-1} \right) (j-l-1) \left(\left\lfloor \frac{j-l-1}{\tilde{k}} \right\rfloor - \frac{j-l-1}{\tilde{k}} + \frac{\tilde{k}-1}{2\tilde{k}} \right) \\ &= \sum_{l=1}^{j-1} l \left(-\frac{l}{k} + \frac{\tilde{k}-1}{2\tilde{k}} \right) - \sum_{l=0}^{\tilde{k}-j} l \left(\frac{l}{k} - \frac{\tilde{k}+1}{2\tilde{k}} \right), \end{aligned}$$

and the result now follows by easy algebraic manipulations.

□

Bibliography

- [1] D. Abramovich, A. Corti, and A. Vistoli. Twisted bundles and admissible covers. *Comm. Algebra*, 31(8):3547–3618, 2003. Special issue in honor of Steven L. Kleiman.
- [2] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.
- [3] D. Abramovich, M. Olsson, and A. Vistoli. Tame stacks in positive characteristic. *Ann. Inst. Fourier (Grenoble)*, 58(4):1057–1091, 2008.
- [4] D. Abramovich and A. Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.*, 15(1):27–75 (electronic), 2002.
- [5] Luis F. Alday, Davide Gaiotto, and Yuji Tachikawa. Liouville correlation functions from four-dimensional gauge theories. *Lett. Math. Phys.*, 91(2):167–197, 2010.
- [6] Luis F. Alday and Yuji Tachikawa. Affine $SL(2)$ conformal blocks from 4d gauge theories. *Lett. Math. Phys.*, 94(1):87–114, 2010.
- [7] J. Alper. Good moduli spaces for Artin stacks. [arXiv:0804.2242 \[math.AG\]](#).
- [8] M. Artin, A. Grothendieck, and J. L. Verdier. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972.
- [9] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.
- [10] Shigetoshi Bando. Einstein-Hermitian metrics on noncompact Kähler manifolds. In *Einstein metrics and Yang-Mills connections (Sanda, 1990)*, volume 145 of *Lecture Notes in Pure and Appl. Math.*, pages 27–33. Dekker, New York, 1993.
- [11] C. Bartocci, U. Bruzzo, and D. Hernández-Ruipérez. *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*. Progress in Mathematics **276**. Birkhäuser, 2009.
- [12] A. Bayer and C. Cadman. Quantum cohomology of $[\mathbb{C}^N/\mu_r]$. *Compos. Math.*, 146(5):1291–1322, 2010.
- [13] K. Behrend and B. Noohi. Uniformization of Deligne-Mumford curves. *J. Reine Angew. Math.*, 599:111–153, 2006.
- [14] V. Belavin and B. Feigin. Super Liouville conformal blocks from $\mathcal{N} = 2$ $SU(2)$ quiver gauge theories. *J. High Energy Phys.*, (7):079, 17, 2011.
- [15] Giulio Bonelli, Kazunobu Maruyoshi, Alessandro Tanzini, and Futoshi Yagi. $\mathcal{N} = 2$ gauge theories on toric singularities, blow-up formulae and W -algebras. *J. High Energy Phys.*, (1):014, front matter + 30, 2013.
- [16] Armand Borel. *Seminar on transformation groups*. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960.
- [17] L. A. Borisov and R. P. Horja. On the K -theory of smooth toric DM stacks. In *Snowbird lectures on string geometry*, volume 401 of *Contemp. Math.*, pages 21–42. Amer. Math. Soc., Providence, RI, 2006.
- [18] Lev A. Borisov, Linda Chen, and Gregory G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.*, 18(1):193–215 (electronic), 2005.
- [19] Niels Borne. Fibrés paraboliques et champ des racines. *Int. Math. Res. Not. IMRN*, (16):Art. ID rnm049, 38, 2007.
- [20] U. Bruzzo and D. Markushevich. Moduli of framed sheaves on projective surfaces. *Documenta Math.*, 16:399–410, 2011.
- [21] U. Bruzzo, D. Markushevich, and A. Tikhomirov. Uhlenbeck-Donaldson compactification for framed sheaves on projective surfaces. *Math. Z.*, , 2013. 10.1007/s00209-013-1170-9.
- [22] U. Bruzzo, M. Pedrini, F. Sala, and R. J. Szabo. Supersymmetric gauge theories on ALE spaces of type A_{k-1} by means of framed sheaves. In preparation.
- [23] U. Bruzzo and F. Sala. Framed sheaves on projective stacks. In preparation.

- [24] Ugo Bruzzo, Francesco Fucito, José F. Morales, and Alessandro Tanzini. Multi-instanton calculus and equivariant cohomology. *J. High Energy Phys.*, (5):054, 24 pp. (electronic), 2003.
- [25] Ugo Bruzzo, Rubik Poghossian, and Alessandro Tanzini. Poincaré polynomial of moduli spaces of framed sheaves on (stacky) Hirzebruch surfaces. *Comm. Math. Phys.*, 304(2):395–409, 2011.
- [26] J. Bryan. A simple identity. <http://www.math.ubc.ca/~jbryan/papers/identity.pdf>.
- [27] C. Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129(2):405–427, 2007.
- [28] E. Carlsson and A. Okounkov. Exts and Vertex Operators. *Duke Math. J.*, 161(9):1797–1815, 2012.
- [29] N. Chriss and V. Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2010. Reprint of the 1997 edition.
- [30] K. Christodouloupoulou. Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras. *J. Algebra*, 320(7):2871–2890, 2008.
- [31] Michele Cirafo, Amir-Kian Kashani-Poor, and Richard J. Szabo. Crystal melting on toric surfaces. *J. Geom. Phys.*, 61(11):2199–2218, 2011.
- [32] D. A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.
- [33] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
- [34] S. K. Donaldson. Instantons and geometric invariant theory. *Comm. Math. Phys.*, 93(4):453–460, 1984.
- [35] D. Edidin and W. Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.
- [36] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli. Brauer groups and quotient stacks. *Amer. J. Math.*, 123(4):761–777, 2001.
- [37] Geir Ellingsrud and Stein Arild Strømme. On the homology of the Hilbert scheme of points in the plane. *Invent. Math.*, 87(2):343–352, 1987.
- [38] Philippe Eyssidieux and Francesco Sala. Instantons and framed sheaves on compact Kähler Deligne-Mumford stacks. In preparation.
- [39] B. Fantechi, E. Mann, and F. Nironi. Smooth toric Deligne-Mumford stacks. *J. Reine Angew. Math.*, 648:201–244, 2010.
- [40] R. Flume and R. Poghossian. Instanton calculations in $N = 2$ super Yang-Mills theory. In *Proceedings of 6th International Workshop on Conformal Field Theory and Integrable Models*, volume May, suppl., pages 168–173, 2004.
- [41] John Fogarty. Algebraic families on an algebraic surface. *Amer. J. Math.*, 90:511–521, 1968.
- [42] I. B. Frenkel and V. G. Kac. Basic representations of affine Lie algebras and dual resonance models. *Invent. Math.*, 62(1):23–66, 1980/81.
- [43] Francesco Fucito, Jose F. Morales, and Rubik Poghossian. Multi-instanton calculus on ALE spaces. *Nuclear Phys. B*, 703(3):518–536, 2004.
- [44] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [45] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [46] D. Gaiotto. Asymptotically free $N=2$ theories and irregular conformal blocks. *ArXiv e-prints*, 2009, 0908.0307.
- [47] E. Gasparim and C.-C. M. Liu. The Nekrasov conjecture for toric surfaces. *Comm. Math. Phys.*, 293(3):661–700, 2010.
- [48] G. Ginot and B. Noohi. Group actions on stacks and applications to equivariant string topology for stacks. 2012, 1206.5603.
- [49] L. Göttsche and W. Soergel. Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. *Math. Ann.*, 296(2):235–245, 1993.
- [50] I. Grojnowski. Instantons and affine algebras. I. The Hilbert scheme and vertex operators. *Math. Res. Lett.*, 3(2):275–291, 1996.
- [51] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.
- [52] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

- [53] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki*, Vol. 6, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.
- [54] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [55] R. Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.
- [56] D. Huybrechts and M. Lehn. Framed modules and their moduli. *Internat. J. Math.*, 6(2):297–324, 1995.
- [57] D. Huybrechts and M. Lehn. Stable pairs on curves and surfaces. *J. Algebraic Geom.*, 4(1):67–104, 1995.
- [58] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.
- [59] H. Iritani. An integral structure in quantum cohomology and mirror symmetry for toric orbifolds. *Adv. Math.*, 222(3):1016–1079, 2009.
- [60] Isamu Iwanari. The category of toric stacks. *Compos. Math.*, 145(3):718–746, 2009.
- [61] Yunfeng Jiang and Hsian-Hua Tseng. The integral (orbifold) Chow ring of toric Deligne-Mumford stacks. *Math. Z.*, 264(1):225–248, 2010.
- [62] V. G. Kac, D. A. Kazhdan, J. Lepowsky, and R. L. Wilson. Realization of the basic representations of the Euclidean Lie algebras. *Adv. in Math.*, 42(1):83–112, 1981.
- [63] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.
- [64] Y. Kawamata. Derived categories of toric varieties. *Michigan Math. J.*, 54(3):517–535, 2006.
- [65] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997.
- [66] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I*. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.
- [67] Sándor J. Kovács. A characterization of rational singularities. *Duke Math. J.*, 102(2):187–191, 2000.
- [68] A. Kresch. On the geometry of Deligne-Mumford stacks. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 259–271. Amer. Math. Soc., Providence, RI, 2009.
- [69] P. B. Kronheimer. The construction of ALE spaces as hyper-Kähler quotients. *J. Differential Geom.*, 29(3):665–683, 1989.
- [70] P. B. Kronheimer and H. Nakajima. Yang-Mills instantons on ALE gravitational instantons. *Math. Ann.*, 288(2):263–307, 1990.
- [71] A. Kuznetsov. Quiver varieties and Hilbert schemes. *Mosc. Math. J.*, 7(4):673–697, 767, 2007.
- [72] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.
- [73] R. Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.
- [74] M. Lehn. *Modulräume gerahmter Vektorbündel*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 241. Universität Bonn Mathematisches Institut, Bonn, 1993. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1992.
- [75] Wei-Ping Li, Zhenbo Qin, and Weiqiang Wang. The cohomology rings of Hilbert schemes via Jack polynomials. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 249–258. Amer. Math. Soc., Providence, RI, 2004.
- [76] A. Licata and A. Savage. A survey of Heisenberg categorification via graphical calculus. *ArXiv e-prints*, 2011, 1104.4789.
- [77] M. Lieblich. Moduli of twisted sheaves. *Duke Math. J.*, 138(1):23–118, 2007.
- [78] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [79] H. Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.
- [80] D. Maulik and A. Okounkov. Quantum Groups and Quantum Cohomology. *ArXiv e-prints*, 2012, 1211.1287.

- [81] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.
- [82] K. Nagao. Quiver varieties and Frenkel-Kac construction. *J. Algebra*, 321(12):3764–3789, 2009.
- [83] H. Nakajima. Jack polynomials and Hilbert schemes of points on surfaces. In *eprint arXiv:alg-geom/9610021*, 1996.
- [84] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.
- [85] H. Nakajima. Sheaves on ALE spaces and quiver varieties. *Mosc. Math. J.*, 7(4):699–722, 767, 2007.
- [86] H. Nakajima and K. Yoshioka. Lectures on instanton counting. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 31–101. Amer. Math. Soc., Providence, RI, 2004.
- [87] H. Nakajima and K. Yoshioka. Instanton counting on blowup. I. 4-dimensional pure gauge theory. *Invent. Math.*, 162(2):313–355, 2005.
- [88] N. A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.*, 7(5):831–864, 2003.
- [89] Nikita A. Nekrasov and Andrei Okounkov. Seiberg-Witten theory and random partitions. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 525–596. Birkhäuser Boston, Boston, MA, 2006.
- [90] F. Nironi. Grothendieck Duality for Deligne-Mumford Stacks. *ArXiv e-prints*, 2008, 0811.1955.
- [91] F. Nironi. Moduli Spaces of Semistable Sheaves on Projective Deligne-Mumford Stacks. *ArXiv e-prints*, 2008, 0811.1949.
- [92] M. Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007.
- [93] M. Olsson and J. Starr. Quot functors for Deligne-Mumford stacks. *Comm. Algebra*, 31(8):4069–4096, 2003. Special issue in honor of Steven L. Kleiman.
- [94] M. C. Olsson. Logarithmic geometry and algebraic stacks. *Ann. Sci. École Norm. Sup. (4)*, 36(5):747–791, 2003.
- [95] Adam Parusiński and Piotr Pragacz. Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors. *J. Amer. Math. Soc.*, 8(4):793–817, 1995.
- [96] M. Pedrini, F. Sala, and R. J. Szabo. AGT on ALE: rank 1 case. In preparation.
- [97] M. Perling. Graded rings and equivariant sheaves on toric varieties. *Math. Nachr.*, 263/264:181–197, 2004.
- [98] Z. Qin and W. Wang. Hilbert schemes of points on the minimal resolution and soliton equations. In *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 435–462. Amer. Math. Soc., Providence, RI, 2007.
- [99] Miles Reid. Canonical 3-folds. In *Journées de Géométrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.
- [100] M. Romagny. Group actions on stacks and applications. *Michigan Math. J.*, 53(1):209–236, 2005.
- [101] O. Schiffmann and E. Vasserot. Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on A^2 . *ArXiv e-prints*, 2012, 1202.2756.
- [102] B. Töen. K-theory and cohomology of algebraic stacks: Riemann-Roch theorems, D-modules and GAGA theorems. [arXiv:math/9908097](https://arxiv.org/abs/math/9908097).
- [103] B. Töen. Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford. *K-Theory*, 18(1):33–76, 1999.
- [104] Hsian-Hua Tseng. Chern classes of Deligne-Mumford stacks and their coarse moduli spaces. *Amer. J. Math.*, 133(1):29–38, 2011.
- [105] Eric Vasserot. Sur l’anneau de cohomologie du schéma de Hilbert de \mathbf{C}^2 . *C. R. Acad. Sci. Paris Sér. I Math.*, 332(1):7–12, 2001.
- [106] A. Vistoli. Rigidification and good moduli space (morphism) in the sense of Alper. MathOverflow. version: 2013-05-15).
- [107] A. Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.*, 97(3):613–670, 1989.
- [108] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [109] Niclas Wyllard. A_{N-1} conformal Toda field theory correlation functions from conformal $\mathcal{N} = 2$ $SU(N)$ quiver gauge theories. *J. High Energy Phys.*, (11):002, 22, 2009.

- [110] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.